A Theoretical Study on the General Circulation of the Pacific Ocean[1]

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INTRODUCTION

The following discussion is one of the results of the research for determining the vertical structure of the wind-driven circulation in an enclosed basin comparable in size to the Pacific Ocean.

The first attempt to verify the effect of prevailing winds in maintaining the oceanic circulation was undertaken by Sverdrup (1947). According to his result, the oceanic currents in the eastern part of the Equatorial Pacific are largely fed by the energy of the winds blowing over the surface of the ocean. Reid (1948) confirmed this conclusion. At nearly the same time, Stommel (1948) could explain the intensification of the wind-driven currents along the west coast of an ocean by assuming the existence of horizontal friction and the meridional variation of Coriolis force. Altogether, these investigations have enabled us to ascribe the major part of the oceanic circulation to the result of the superincumbent wind system prevailing over the oceans.

Munk (1950) published a very important paper on the wind-driven circulation of the oceans. The next year Munk and Carrier (1951) treated the circulation of the North Pacific, regarding this ocean as a triangular basin. They could explain the pattern of the actual ocean circulation very well, so that there seems to be little left to discuss on this subject, as far as the major characteristics of the general circulation in the Pacific Ocean are concerned.

Hidaka (1951) solved the problem of the general circulation which would be produced by both zonal and anticyclonic wind systems. In this computation, spherical co-ordinates were used, thus taking the effect of the sphericity of the earth into account. But both assumptions of the wind distribution gave no essential difference in the results except for the magnitude of mass transport. Moreover, the result for zonal distribution gave no sensible difference in the pattern of the circulation compared with Munk's which was derived by using a rectangular co-ordinate system, except in the magnitude of the mass transport. These facts show us that we have only to treat the circulation driven by a zonal wind system.

An earlier paper (Hidaka, 1950) contained a theory of ocean circulation using the current velocity in place of the mass transport. The analysis was complex because the vertical variation of the movement of water had to be taken into account. The result was, nonetheless, quite ridiculous; no perceptible concentration of the streamlines toward the west coast could be found. The explanation for this result was that the approximation to the

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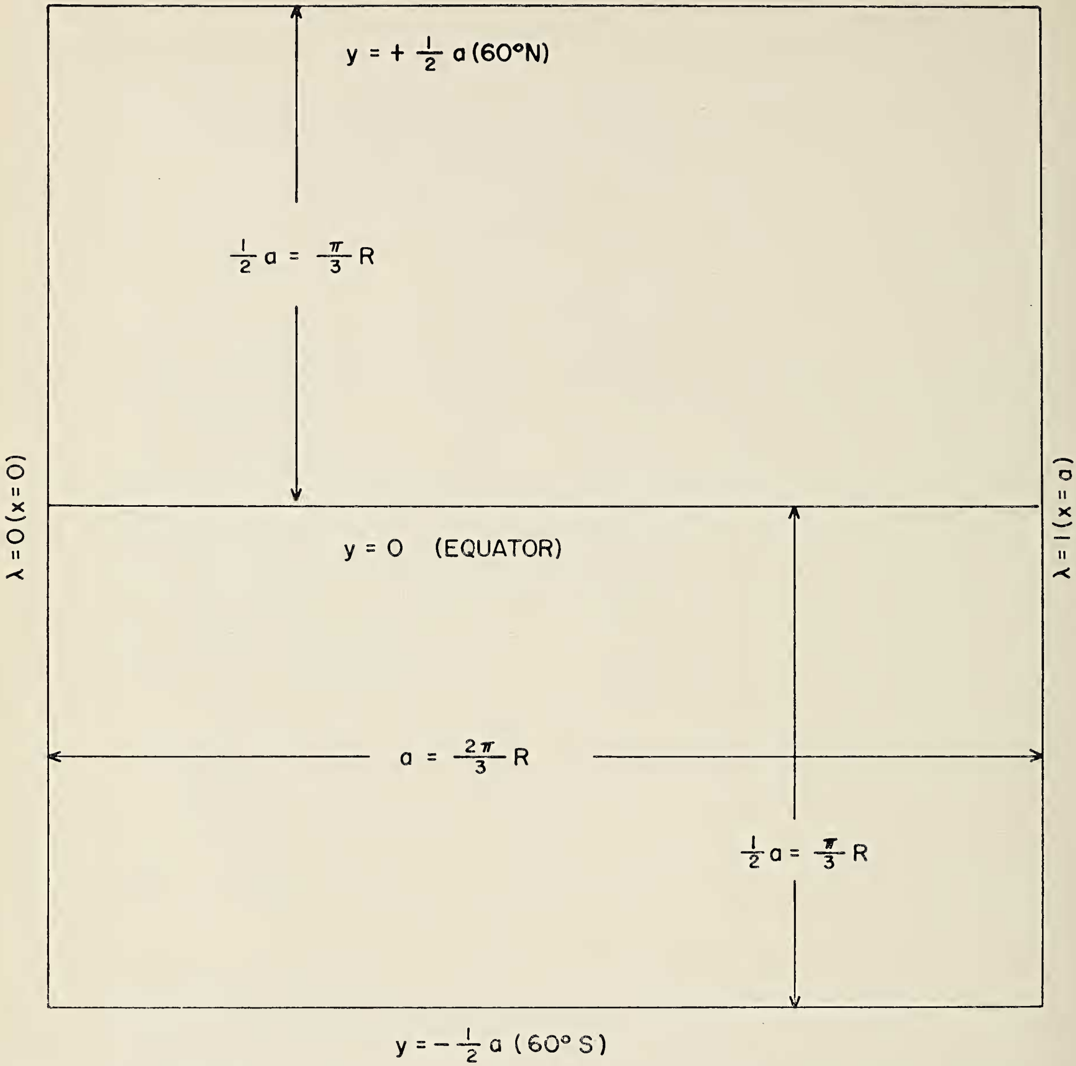


FIG. 1. A rectangular ocean comparable in size to the Pacific Ocean.

mass transport was inadequate for its east-west variation. Thus, the solution smoothed out the western currents and boundary vortices which were apparent in Munk's paper.

The mass transport method which has been adopted by Defant (1941), Stockmann (1945–46), Sverdrup (1947), Reid (1948), Munk (1950, 1951), and recently by Hansen (1952) is surely eminent, especially in that it enables us to reduce the analysis to two dimensions and makes the mathematical procedure very simple. Moreover, it is not necessary for us to consider the vertical variation of density and vertical coefficient of eddy viscosity. These authors have indeed contributed greatly to the solution of many important problems of oceanic circulation by this method. The author himself also employed this method several times in discussing the problems in this direction. However, it is impossible for this method to show how the wind-driven circulation varies in a vertical direction. Neumann also expressed recently (1952) his opinion as to the necessity of a dynamic treatment of ocean currents as a three-dimensional problem.

All these circumstances lead one to recompute the general circulation of the Pacific Ocean under these modified conditions and assumptions. The present investigation is one of the results of the author's efforts in this direction. We here treat the general circulation of the water in a square ocean comparable in size to the entire Pacific Ocean basin. Spherical co-ordinates, which were used in a preceding paper (Hidaka, 1951), are not used here, partly in order to avoid mathematical difficulties but mostly because the two systems of co-ordinates did not give any important difference between Munk's and the author's results except for the magnitude of mass transport. The value of the lateral mixing coefficient is taken as 3.08×10^7 c.g.s., a value consistent with the research of former investigators. The wind system is considered zonal, because this assumption is far simpler for the subsequent analysis, and also because no essential difference has been found between the results obtained under the assumptions of zonal and anticyclonic wind distributions. Of course, the variation of the Coriolis parameter with latitude is taken into account. The use of current velocity in place of mass transport makes the mathematical analysis many times more complicated because the problem is now three dimensional. But the result will be of importance because it should give an idea of the vertical structure of the wind-driven circulation of the oceans.

THEORY

The dynamic equations of the stationary ocean currents, taking both vertical and horizontal mixing into account and neglecting the nonlinear terms, are

$$\begin{aligned} A_l \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial z}\left(A_z \frac{\partial u}{\partial z}\right) + 2\omega \sin \phi \rho v - \frac{\partial p}{\partial x} &= 0, \\ A_l \frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial z}\left(A_z \frac{\partial v}{\partial z}\right) - 2\omega \sin \phi \rho u - \frac{\partial p}{\partial y} &= 0 \end{aligned} \tag{1}$$

where u and v are components of the current velocity in x (eastward positive) and y (northward positive) directions, p is the pressure, ρ the density, A_l and A_z the horizontal and vertical coefficients of eddy viscosity of the water, ω the angular velocity of the earth, and ϕ the geographical latitude. The axis of z is taken positive downward, the origin being placed on the undisturbed sea level.

The boundary conditions to be satisfied on the surface ($z = 0$) and at the bottom ($z = h$) are

$$z = 0: -A_z \frac{\partial u}{\partial z} = \tau_x(x, y); \; -A_z \frac{\partial v}{\partial z} = \tau_y(x, y) \qquad (2$$

and

$$z = h: u = v = 0. \tag{3}$$

Here both coefficients of eddy viscosity are supposed to be constants. The conditions to be satisfied along the coasts are also necessary. These are simply that there is no water flow across and along these coasts. If the coasts consist of vertical cliffs, we have

$$u = v = 0 \tag{4}$$

at the shore lines.

In addition to the dynamic equations (1), the equation of continuity should be included. If we neglect the vertical component of velocity, it is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5}$$

Equation (5) assumes that there is neither vertical current nor vertical gradient of the vertical velocity. Thus, our theory cannot be applied to the coastal and other regions of upwelling and sinking caused by local monsoons or other temporary winds. But, if we confine ourselves to the gross features of the horizontal circulation in great oceans, induced by the superincumbent, quasi-permanent wind system (westerlies or trades), the equation of continuity as given by (5) will not cause serious errors in the results.

There may be some further question concerning the use of (5) for the continuity equation. But, in treating the oceanographic data for estimating geostrophic currents, we always assume

$$u = + \frac{1}{2\omega \sin \phi\rho} \frac{\partial p}{\partial y},$$

$$v = - \frac{1}{2\omega \sin \phi\rho} \frac{\partial p}{\partial x},$$

provided the frictional terms are neglected. These expressions imply that geostrophic currents usually satisfy equation (5) which shows the absence of horizontal divergence. This means that the equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ may be used without serious errors in treating the general circulation of the oceans.

On eliminating p between the two equations of (1) by cross-differentiation, we have

$$A_l \left(\frac{\partial^3 u}{\partial y^3} - \frac{\partial^3 v}{\partial x^3}\right) + \frac{\partial}{\partial z}\left\{A_z \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial z}\right)\right\} + 2\omega\rho \frac{d}{dy}(\sin \phi)\, v = 0 \tag{6}$$

when we take the equation of continuity (5) into account and because $\frac{\partial}{\partial x}(\sin \phi) = 0$. This equation may also be regarded as expressing the condition that a function p (pressure) should exist on a level z as an exact differential with respect to x and y. And the validity of equation (6) suggests the possibility of determining the pressure in any level z as a function of x and y.

Now suppose the coefficients of mixing A_l and A_z are both independent of z, aud put

$$u = \sum_{s=1}^{\infty} u_s(x, y) \cos \frac{(2s-1)\pi z}{2h}, \tag{7}$$

where

$$u_s(x, y) = \frac{2}{h}\int_0^h u(x, y, \zeta) \cos \frac{(2s-1)\pi\zeta}{2h} d\zeta \tag{8}$$

and assume, in accord with Stokes,

$$\frac{\partial^2 u}{\partial z^2} = \frac{2}{h}\sum_{s=1}^{\infty} \cos \frac{(2s-1)\pi z}{2\text{h}} \int_0^h \frac{\partial^2 u}{\partial \zeta^2} \cos \frac{(2s-1)\pi\zeta}{2\text{h}} d\zeta; \tag{9}$$

then we have, by substitution from (2), (3), and (8),

$$\frac{2}{h}\int_0^h \frac{\partial^2 u}{\partial \zeta^2}\cos\frac{(2s-1)\pi\zeta}{2h}\,d\zeta = \frac{2}{h}\frac{\tau_x(x,y)}{A_z} - \frac{(2s-1)^2\pi}{4h^2}u_s(x,y)$$

and (9) becomes

$$\frac{\partial^2 u}{\partial z^2} = \sum_{s=1}^{\infty}\left\{\frac{2}{h}\frac{\tau_x(x,y)}{A_z} - \frac{(2s-1)^2\pi^2}{4h^2}u_s(x,y)\right\}\cos\frac{(2s-1)\pi z}{2h} \tag{10}$$

Similarly we have

$$\frac{\partial^2 v}{\partial z^2} = \sum_{s=1}^{\infty}\left\{\frac{2}{h}\frac{\tau_y(x,y)}{A_z} - \frac{(2s-1)^2\pi^2}{4h^2}v_s(x,y)\right\}\cos\frac{(2s-1)\pi z}{2h}, \tag{11}$$

where

$$v = \sum_{s=1}^{\infty} v_s(x,y)\cos\frac{(2s-1)\pi z}{2h}. \tag{12}$$

Substitutions of (7), (10), (11), and (12) into (6) and (5) give

$$A_l\left(\frac{\partial^3 u_s}{\partial y^3} - \frac{\partial^3 v_s}{\partial x^3}\right) - \frac{(2s-1)^2\pi^2 A_z}{4h^2}\left(\frac{\partial u_s}{\partial y} - \frac{\partial v_s}{\partial x}\right) + 2\omega\rho\frac{\cos\phi}{R}v_s + \frac{2}{h}\left(\frac{\partial \tau_x}{\partial y} - \frac{\partial \tau_y}{\partial x}\right) = 0 \tag{13}$$

where R is the radius of the earth, and

$$\frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial y} = 0. \tag{14}$$

Equation (14) gives a set of functions $\psi_s\ (x, y)$ such that

$$u_s = \frac{\partial \psi_s}{\partial y};\ v_s = -\frac{\partial \psi_s}{\partial x} \tag{15}$$

and (13) becomes

$$A_l\left(\frac{\partial^4 \psi}{\partial x^4} + \frac{\partial^4 \psi_s}{\partial y^4}\right) - \frac{(2s-1)^2\pi^2 A_z}{4h^2}\left(\frac{\partial^2 \psi_s}{\partial x^2} + \frac{\partial^2 \psi_s}{\partial y^2}\right) - 2\,\omega\rho\frac{\cos\phi}{R}\frac{\partial \psi_s}{\partial x} + \frac{2}{h}\left(\frac{\partial \tau_x}{\partial y} - \frac{\partial \tau_y}{\partial x}\right) = 0. \tag{16}$$

If we introduce two quantities D_l and D_z such that

$$D_l = \pi\sqrt{\frac{A_l}{\rho\omega}} \tag{17}$$

and

$$D_z = \pi\sqrt{\frac{A_z}{\rho\omega}}, \tag{18}$$

equation (16) becomes

$$\frac{D_l^2}{2\pi^2}\left(\frac{\partial^4 \psi}{\partial x^4} + \frac{\partial^4 \psi}{\partial y^4}\right) - \frac{(2s-1)^2}{8}\left(\frac{D_z}{h}\right)^2\left(\frac{\partial^2 \psi_s}{\partial x^2} + \frac{\partial^2 \psi_s}{\partial y^2}\right) - \frac{\cos\phi}{R}\frac{\partial \psi_s}{\partial x} + \frac{1}{\rho\omega h}\left(\frac{\partial \tau_y}{\partial y} - \frac{\partial \tau_x}{\partial x}\right) = 0. \tag{19}$$

The quantity D_l may be called "the frictional distance," whereas D_z is the same as Ekman's "depth of frictional influence" except that it does not contain $\sin\phi$.

The coastal conditions which ψ_s should satisfy are

$$\psi_s \text{ and } \frac{\partial \psi_s}{\partial n} = 0 \tag{20}$$

along the shore lines, where $\frac{\partial \psi_s}{\partial n}$ is the derivative of ψ_s in the direction normal to the shore lines.

If equation (19) can be solved and we can determine the functions $\psi_1(x, y)$, $\psi_2(x, y)$, $\psi_3(x, y)$, . . . , the sum:

$$\psi(x, y, z) = \sum_{s=1}^{\infty} \psi_s(x, y) \cos \frac{(2s-1)\pi z}{2h} \tag{21}$$

will give the horizontal streamlines at any level z for the wind stresses $\tau_x(x, y)$ and $\tau_y(x, y)$. The stream function $\psi(x, y, z)$ should, of course, satisfy the condition:

$$\psi = \frac{\partial \psi}{\partial n} = 0 \tag{22}$$

along the shore lines and the horizontal streamlines of the currents are given by

$$\psi(x, y, z) = \text{constant}. \tag{23}$$

Infinitely Deep Ocean

If ψ_s^1 is the solution of (19) when $h = 1$, the solution of (19) will be

$$\psi_s(x, y) = \frac{1}{h} \psi_s^1(x, y). \tag{24}$$

Thus we have the solution (21) of our problem in the form:

$$\psi(x, y, z) = \sum_{s=1}^{\infty} \psi_s^1(x, y) \frac{1}{h} \cos \frac{(2s-1)\pi z}{2h}, \tag{25}$$

and we may write down (25) in the form:

$$\psi(x, y, z) = \sum_{s=1}^{\infty} \frac{\psi_s^1(x, y)}{2D_z} \cos \left\{ \frac{\pi z}{2D_z} (2s-1) \frac{D_z}{h} \right\} \frac{2D_z}{h}.$$

If we consider the depth h of the ocean increases indefinitely, we have

$$(2s-1)\frac{D_z}{h} = \eta, \quad \frac{2D_z}{h} = d\eta \tag{26}$$

and

$$\psi^1(x, y) \rightarrow \psi^1(x, y; \eta), \tag{27}$$

where $\psi_1(x, y, \eta)$ is the solution of the equation:

$$\frac{A_l}{2\omega\rho}\left(\frac{\partial^4 \psi_1}{\partial x^4} + \frac{\partial^4 \psi_1}{\partial y^4}\right) - \frac{\eta^2}{8}\left(\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2}\right) - \frac{\cos\phi}{R}\frac{\partial \psi_1}{\partial x} + \frac{1}{\rho\omega}\left(\frac{\partial \tau_x}{\partial y} - \frac{\partial \tau_y}{\partial x}\right) = 0 \tag{28}$$

which is derived from (19) by putting $(2s-1)\frac{D_z}{h} = \eta$ and $h = 1$. The right-hand sides of equation (24) will be, when the depth h increases indefinitely,

$$\psi(x, y, z) = \frac{1}{2D_z} \int_0^{\infty} \psi_1(x, y; \eta) \cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta. \tag{29}$$

It may be mentioned that η is a parameter increasing from 0 to ∞.

APPLICATION TO THE PACIFIC OCEAN

In 1950, Munk used the rectangular co-ordinates in discussing the wind-driven oceanic circulation in a rectangular ocean. Though he did not take the sphericity of the earth into consideration, he could explain the general pattern of the actual circulation of the Pacific quite well. Further, his result shows little difference from the one above (Hidaka, 1951), in which the curvature of the surface of the earth is taken into account and spherical co-ordinates are used.

These results show that the Pacific Ocean can be treated approximately as a rectangular ocean, provided we consider that the Coriolis parameter $2\omega \sin \phi$ varies with y. This is a quite natural consequence when we consider the relation $\phi = \frac{y}{R}$, where R is the mean radius of the earth, and y is counted zero on the equator.

For these reasons, we can represent the Pacific approximately as a square ocean bounded by $x = 0$, $x = a$, and $y = \pm\frac{a}{2}$, in which $y = 0$ coincides with the equator (Figure 1). Here a is a mean east–west extent of this ocean and approximately equals 120° of longitude or $2\pi/3$ in radians. This means that the northern and southern boundaries of this ocean are given by $y = \pm\frac{2\pi R}{3}$ or the parallels of 60° N and S.

To solve equation (28), assume

$$\psi_1(x, y; \eta) = \sum_m M_m(y)\, N_m(x; \eta), \tag{30}$$

where $m = 1, 2, 3, \ldots$ and $N_m\,(x, \eta)$ are functions of x which are to be determined later, while $M_m\,(y)$ are of the forms:

$$\begin{aligned} M_m(y) &= \cos\frac{\pi y}{2a}\cos\frac{m\pi y}{2a} \qquad \text{for } m\text{: odd} \\ &= \cos\frac{\pi y}{2a}\sin\frac{m\pi y}{2a} \qquad \text{for } m\text{: even.} \end{aligned} \tag{31}$$

Since these functions and their y-derivatives vanish along $y = \pm\frac{a}{2}$, (30) and its y-derivatives will also vanish along $y = \pm\frac{a}{2}$. This makes the function $\psi\,(x, y, z)$ satisfy the condition (22) along the northern and southern boundaries $y = \pm\frac{a}{2}$.

Stresses of Winds over the Pacific Ocean

Dr. Munk kindly furnished me with his unpublished data of the distribution of the wind stresses over the Pacific Ocean north of 5° S. From these data, the most probable

TABLE I
ASSUMED MERIDIONAL VARIATION OF THE WIND STRESS τ AND STRESS FUNCTION T

y/a	LATITUDE	T *dynes/cm*	τ *dyne/cm²*	REMARKS	y/a	LATITUDE	T *dynes/cm*	τ *dyne/cm²*	REMARKS
+ .500	60° N	.000	.000		0.000	0°	+ .201 × 10⁸	−.272	
+ .475	57° N	− .037 × 10⁸	.211	West	−0.025	3° S	.304 × 10⁸	−.360	SE
+ .450	54° N	− .141 × 10⁸	.396		−0.050	6° S	.441 × 10⁸	−.463	
+ .425	51° N	− .301 × 10⁸	.528	Wind	−0.075	9° S	.663 × 10⁸	−.554	Trades
+ .400	48° N	− .493 × 10⁸	.589		−0.100	12° S	.811 × 10⁸	−.606	
+ .375	45° N	− .694 × 10⁸	.573	Drift	−0.125	15° S	1.018 × 10⁸	−.595	
+ .350	42° N	− .876 × 10⁸	.482		−0.150	18° S	1.208 × 10⁸	−.514	
+ .325	39° N	−1.015 × 10⁸	.327		−0.175	21° S	1.360 × 10⁸	−.369	
+ .300	36° N	−1.094 × 10⁸	.132		−0.200	24° S	1.454 × 10⁸	−.174	
+ .275	33° N	−1.104 × 10⁸	−.075		−0.225	27° S	1.476 × 10⁸	+.045	
+ .250	30° N	−1.044 × 10⁸	−.266		−0.250	30° S	1.424 × 10⁸	+.259	
+ .225	27° N	− .926 × 10⁸	−.416	NE	−0.275	33° S	1.303 × 10⁸	+.438	West
+ .200	24° N	− .767 × 10⁸	−.507		−0.300	36° S	1.131 × 10⁸	+.564	
+ .175	21° N	− .588 × 10⁸	−.534	Trades	−0.325	39° S	.927 × 10⁸	+.626	Wind
+ .150	18° N	− .411 × 10⁸	−.502		−0.350	42° S	.714 × 10⁸	+.626	
+ .125	15° N	− .253 × 10⁸	−.431		−0.375	45° S	.510 × 10⁸	+.574	Drift
+ .100	12° N	− .123 × 10⁸	−.343		−0.400	48° S	.331 × 10⁸	+.484	
+ .075	9° N	− .024 × 10⁸	−.264		−0.425	51° S	.187 × 10⁸	+.371	
+ .050	6° N	+ .053 × 10⁸	−.219	Doldrum	−0.450	54° S	.083 × 10⁸	+.249	
+ .025	3° N	+ .122 × 10⁸	−.219		−0.475	57° S	.021 × 10⁸	+.124	
+ .000	0°	+ .201 × 10⁸	−.272		− .500	60° S	.000 × 10⁸	.000	

distribution of the wind stresses was found to be zonal and determined as

$$\begin{aligned}\tau_x(\phi) = {} & + 0.045544\, M_1'(\phi) - 0.262308\, M_2'(\phi) \\ & + 0.022902\, M_3'(\phi) + 0.069493\, M_4'(\phi) \\ & - 0.036900\, M_5'(\phi) + 0.011560\, M_6'(\phi)\ \text{dyne/cm}^2\end{aligned} \tag{32}$$

where ϕ is latitude so that $\phi = y/R$ and $M_m'(\phi)$ stand for $\frac{d}{d\phi} M_m(\phi) = R \frac{d}{dy} M_m(\phi)$.

If we are to derive (32) from a stress function T (x, y) such that

$$\tau_x = \frac{\partial \mathrm{T}}{\partial y};\ \tau_y = -\frac{\partial \mathrm{T}}{\partial x}, \tag{33}$$

we have a stress function which depends on ϕ or y only, or

$$\begin{aligned}\mathrm{T}(\phi) &= \sum_{m=1}^{\infty} \mathrm{T}_m \cdot M_m(\phi) \\ &= \{ 0.29003\, M_1(\phi) - 1.67122\, M_2(\phi) \\ &\quad + 0.14591\, M_3(\phi) + 0.44276\, M_4(\phi) \\ &\quad - 0.23510\, M_5(\phi) + 0.07388\, M_6(\phi) \} \\ &\quad \times 10^8 \text{ dynes/cm},\end{aligned} \tag{34}$$

provided we take $R = 6.3712 \times 10^8$ cm.

The values of stress and stress function computed after the formulas (32) and (34) are compiled in Table 1 and illustrated in Figure 2.

From formula (32) we have a much larger wind stress in the west wind belt in the South Pacific Ocean than in the North Pacific. It is not because we had wind observations available from the former, but is a necessary consequence resulting from determining the formula so as to meet the observational data for the latitudes north of 5° S. As a matter of fact, the west wind belt in the South Pacific is said to have stronger wind than that of the North Pacific. So this formula will not give wind stresses in the South Pacific too inconsistent with observations.

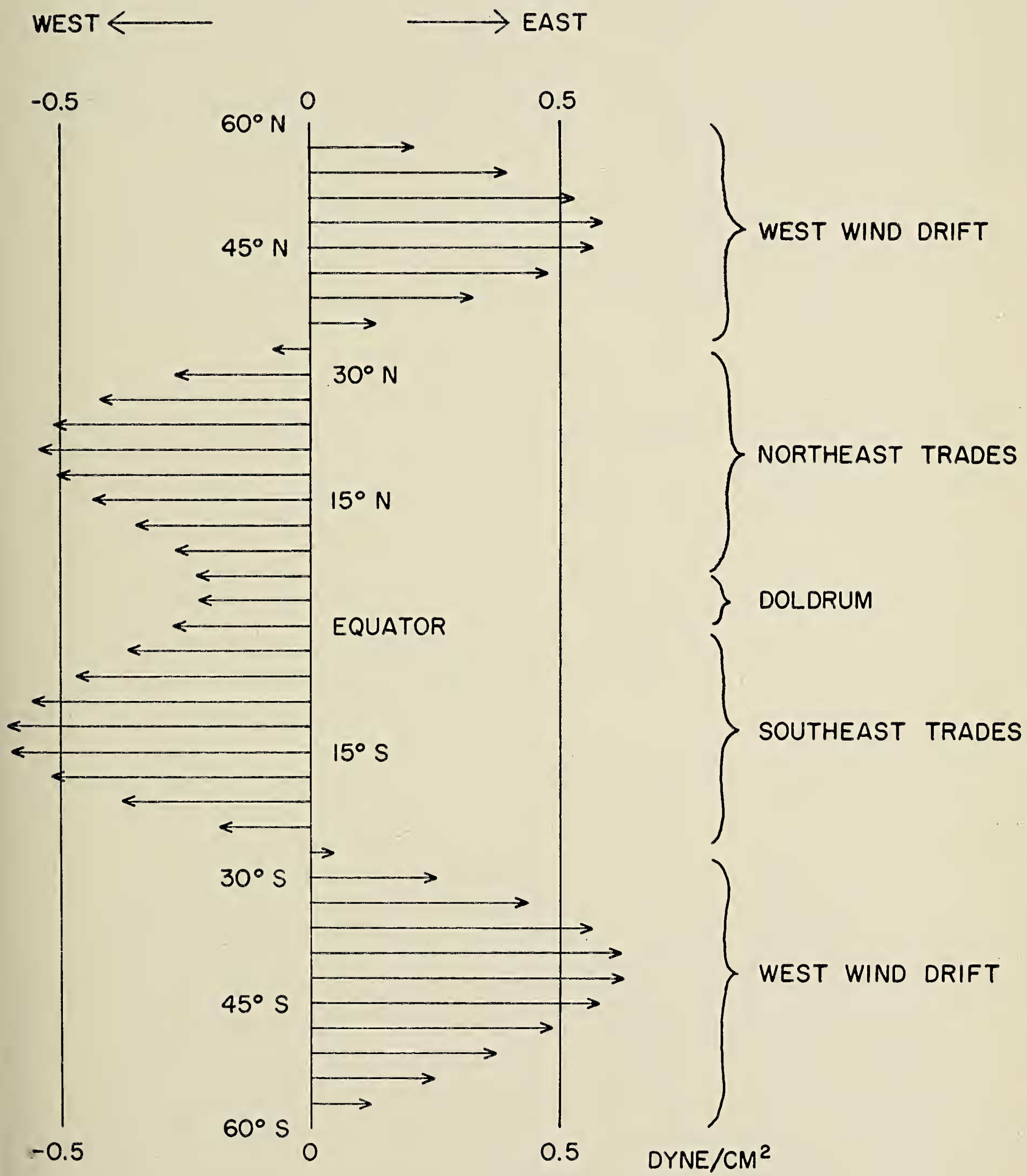


FIG. 2. Assumed meridional distribution of wind stress.

Elimination of y-Co-ordinate

The equation to be solved is from (28)

$$\frac{D_l^2}{2\pi^2}\left(\frac{\partial^4\psi_1}{\partial x^4}+\frac{\partial^4\psi_1}{\partial y^4}\right)-\frac{\eta^2}{8}\left(\frac{\partial^2\psi_1}{\partial x^2}+\frac{\partial^2\psi_1}{\partial y^2}\right)-\frac{\cos\phi}{R}\frac{\partial\psi_1}{\partial x}+\frac{2}{\rho\omega}\left(\frac{\partial\tau_x}{\partial y}-\frac{\partial\tau_y}{\partial x}\right)=0 \tag{35}$$

where

$$D_l = \pi\sqrt{\frac{A_l}{\rho\omega}}. \tag{36}$$

Between the equator and 60° N or S, cos ϕ varies from 1 to ½. Since this is not a large variation, we may take the average value for cos ϕ for this range, or

$$\cos\phi = \frac{1}{2\pi/3}\int_{-\frac{\pi}{3}}^{+\frac{\pi}{3}}\cos\phi d\phi = \frac{3\sqrt{3}}{2\pi}.$$

Thus, the third term in (35) now becomes approximately

$$-\frac{\cos\phi}{R}\frac{\partial\psi_1}{\partial x} = -\frac{3\sqrt{3}}{2\pi R}\frac{\partial\psi_1}{\partial x}. \tag{37}$$

Substituting (30), (32), (34), and (37) in (35), we have

$$\begin{aligned}\sum_m\Bigg[\frac{D_l^2}{2\pi^2}&\left\{\frac{1}{(\frac{2\pi}{3})^4R^4}\cdot\frac{d^4N_m}{d\lambda^4}\quad M_m(\phi)+N_m(\lambda;\eta)\frac{1}{R^4}\cdot\frac{d^4N_m}{d\phi^4}\right\}\\ &-\frac{\eta^2}{8}\left\{\frac{1}{(\frac{2\pi}{3})^2R^2}\cdot\frac{d^2N_m}{d\lambda^2}\cdot M_m(\phi)+N_m(\lambda;\eta)\frac{1}{R^2}\cdot\frac{d^2M_m}{d\phi^2}\right\}\\ &+N_m(\lambda;\eta)\cdot\frac{1}{R^2}\cdot\frac{d^2M_m}{d\phi^2}-\frac{\frac{3\sqrt{3}}{2\pi}}{\frac{2\pi}{3}R^2}\cdot\frac{dN_m}{d\lambda^2}\cdot M_m(\phi)\Bigg]\\ &-\frac{1}{\rho\omega}\cdot\frac{1}{R^2}\sum_m \mathrm{T}_m\cdot\frac{d^2M_m}{d\phi^2}=0\end{aligned} \tag{38}$$

where we have

$$\lambda = \frac{x}{(2\pi R/3)}, \tag{39}$$

so that the ocean is supposed to be bounded by the meridians of 0° and 120°, and the western and eastern boundaries are given by $\lambda = 0$ and $\lambda = 1$ respectively.

Now it is possible to express $\frac{d^2M_m}{dy^2}$ and $\frac{d^4M_m}{dy^4}$ in terms of $M_m(\phi)$, or

$$\frac{d^2M_m}{dy^2} = \sum_i \alpha_i^m M_i(\phi)$$

and (40)

$$\frac{d^4M_m}{dy^4} = \sum_i \beta_i^m M_i(\phi).$$

Substituting (40) in (38) we have

$$\sum_m M_m(\phi) \cdot \left\{ \frac{D_l^2}{2\pi^2} \cdot \frac{81}{16\pi^4 R^4} \cdot \frac{d^4 N_m}{d\lambda^4} - \frac{\eta^2}{8} \cdot \frac{9}{4\pi^2 R^2} \cdot \frac{d^2 N_m}{dy^2} - \frac{\frac{3\sqrt{3}}{2\pi}}{\frac{2\pi}{3} R} \cdot \frac{dN_m}{d\lambda} \right.$$

$$\left. + \sum_i \left(- \frac{\eta^2}{8} \cdot \frac{1}{R^2} \cdot \alpha_m^i + \frac{D_l^2}{2\pi^2} \cdot \frac{1}{R^4} \cdot \beta_m^i \right) N_i + \frac{1}{\rho\omega} \cdot \frac{1}{R^2} \sum \alpha_m^i \mathrm{T}_i \right\} = 0. \quad (41)$$

In order that this relation should be always valid, the coefficients of $M_m(\phi)$ must vanish. Thus, we have

$$\tfrac{81}{32} \frac{D_l^2}{R^2} \frac{d^4 N_m}{d\lambda^4} - \frac{9}{32\pi^2} \eta^2 \frac{d^2 N_m}{d\lambda^2} - \frac{9\sqrt{3}}{4\pi^2} \frac{dN_m}{d\lambda} + \sum_i \left(- \tfrac{1}{8} \alpha_m^i \eta^2 + \frac{1}{2\pi^2} \beta_m^i \frac{D_l^2}{R^2} \right) N_i + \frac{1}{\rho\omega} \sum_i \alpha_m^i \mathrm{T}_i = 0 \quad (42)$$

where T_i is given by (34).

It may be anticipated from (39) and (40) that the values of β_m^i are larger than that of α_m^i for any corresponding set of values of i and m. But the ratio

$$\frac{D_l^2}{R^2} = \frac{\pi^2}{\rho\omega} \frac{A_l}{R^2}$$

is of the order of 10^{-4} for the usual value of A_l $(= 10^8)$, so that we may neglect the terms of β_m^i in the coefficients of N_i in (42). Then (42) becomes

$$\tfrac{81}{32} \frac{D_l^2}{R^2} \cdot \frac{d^4 N_m}{d\lambda^4} - \frac{9}{32\pi^2} \eta^2 \frac{d^2 N_m}{d\lambda^2} - \frac{9\sqrt{3}}{4\pi^2} \frac{dN_m}{d\lambda} - \frac{\eta^2}{8} \sum_i \alpha_m^i N_i + \frac{1}{\rho\omega} \sum_i \alpha_m^i \mathrm{T}_i = 0. \quad (43)$$

Determination of the Values of α_m^i

The coefficients of $M_m(\phi)$ in the expansions of $\dfrac{d^2 M_m}{d\phi^2}$ were evaluated as far as $M_{10}(\phi)$, the higher terms being neglected.

For the ocean under consideration, we have

$$\begin{aligned}
M_1''(\phi) &= -\tfrac{9}{11}\{M_1(\phi) + 8M_3(\phi) - 6M_5(\phi) + 4M_7(\phi) - 2M_9(\phi)\},\\
M_3''(\phi) &= -\tfrac{9}{11}\{-3M_1(\phi) + 20M_2(\phi) + 18M_5(\phi) - 12M_7(\phi) + 6M_9(\phi)\},\\
M_5''(\phi) &= -\tfrac{9}{11}\{+5M_1(\phi) - 15M_3(\phi) + 69M_5(\phi) + 20M_7(\phi) - 10M_9(\phi)\},\\
M_7''(\phi) &= -\tfrac{9}{11}\{-7M_1(\phi) + 21M_3(\phi) - 35M_5(\phi) + 148M_7(\phi) + 14M_9(\phi)\},\\
M_9''(\phi) &= -\tfrac{9}{11}\{+9M_1(\phi) - 27M_3(\phi) + 45M_5(\phi) - 63M_7(\phi) + 257M_9(\phi)\}
\end{aligned} \quad (44)$$

and

$$\begin{aligned}
M_2''(\phi) &= -\tfrac{3}{4}\{7M_2(\phi) + 16M_4(\phi) - 12M_6(\phi) + 8M_8(\phi) - 4M_{10}(\phi)\},\\
M_4''(\phi) &= -\tfrac{3}{4}\{-8M_2(\phi) + 43M_4(\phi) + 24M_6(\phi) - 16M_8(\phi) + 8M_{10}(\phi)\},\\
M_6''(\phi) &= -\tfrac{3}{4}\{+12M_2(\phi) - 24M_4(\phi) + 111M_6(\phi) + 24M_8(\phi) - 12M_{10}(\phi)\},\\
M_8''(\phi) &= -\tfrac{3}{4}\{-16M_2(\phi) + 32M_4(\phi) - 48M_6(\phi) + 211M_8(\phi) + 16M_{10}(\phi)\},\\
M_{10}''(\phi) &= -\tfrac{3}{4}\{20M_2(\phi) - 40M_4(\phi) + 60M_6(\phi) - 80M_8(\phi) + 343M_{10}(\phi)\}.
\end{aligned} \quad (45)$$

Substituting (44) and (45) into equation (43), and evaluating the numerical values of their coefficients with $R = 6.3712 \times 10^8$ cm., $\rho\omega = 0.000075$ sec^{-1} and

$$A_1 = 3.08 \times 10^7 \text{ c.g.s.}, \quad (46)$$

we have for odd series

$$
\begin{aligned}
&D(N_1) + 4747600\eta^2 \times \tfrac{9}{11}(N_1 - 3N_3 + 5N_5 - 7N_7 + 9N_9) \\
&\qquad - 0.548249 \times 10^{20} = 0, \\
&D(N_3) + 4747600\eta^2 \times \tfrac{9}{11}(8N_1 + 20N_3 - 15N_5 + 21N_7 - 27N_9) \\
&\qquad - 3.63163 \times 10^{20} = 0, \\
&D(N_5) + 4747600\eta^2 \times \tfrac{9}{11}(-6N_1 + 18N_3 + 69N_5 - 35N_7 + 45N_9) \\
&\qquad + 6.35413 \times 10^{20} = 0, \\
&D(N_7) + 4747600\eta^2 \times \tfrac{9}{11}(+4N_1 - 12N_3 + 20N_5 + 148N_7 - 63N_9) \\
&\qquad + 2.19300 \times 10^{20} = 0, \\
&D(N_9) + 4747600\eta^2 \times \tfrac{9}{11}(-2N_1 + 6N_3 - 10N_5 + 14N_7 + 257N_9) \\
&\qquad - 1.096499 \times 10^{20} = 0,
\end{aligned} \tag{47}
$$

where the operator D stands for

$$
D = \frac{d^4}{d\lambda^4} - 1082323\eta^2 \frac{d^2}{d\lambda^2} - 14997110 \frac{d}{d\lambda}. \qquad 48)
$$

For even series, we have

$$
\begin{aligned}
&D(N_2) + 4747600\eta^2 \times \tfrac{3}{4}(+7N_2 - 8N_4 + 12N_6 - 16N_8 \\
&\qquad + 20N_{10}) + 5.45179 \times 10^{20} = 0, \\
&D(N_4) + 4747600\eta^2 \times \tfrac{3}{4}(+16N_2 + 43N_4 - 24N_6 + 32N_8 \\
&\qquad - 40N_{10}) + 3.59829 \times 10^{20} = 0, \\
&D(N_6) + 4747600\eta^2 \times \tfrac{3}{4}(-12N_2 + 24N_4 + 111N_6 - 48N_8 \\
&\qquad + 60N_{10}) - 14.76753 \times 10^{20} = 0, \\
&D(N_8) + 4747600\eta^2 \times \tfrac{3}{4}(+8N_2 - 16N_4 + 24N_6 + 211N_8 \\
&\qquad - 80N_{10}) + 7.09512 \times 10^{20} = 0, \\
&D(N_{10}) + 4747600\eta^2 \times \tfrac{3}{4}(-4N_2 + 8N_4 - 12N_6 + 16N_8 \\
&\qquad + 343N_{10}) - 3.54756 \times 10^{20} = 0,
\end{aligned} \tag{49}
$$

where the operator D is also given by (48).

To solve the simultaneous differential equations (47) and (48), we employ the method of indeterminate multipliers. Let the odd set of equations be of the forms:

$$
\begin{aligned}
D(N_1) + \sum_{i=1}^{9} C_i^1 N_i + E_1 &= 0, \\
D(N_3) + \sum_{i=1}^{9} C_i^3 N_i + E_3 &= 0, \\
D(N_5) + \sum_{i=1}^{9} C_i^5 N_i + E_5 &= 0, \\
D(N_7) + \sum_{i=1}^{9} C_i^7 N_i + E_7 &= 0, \\
D(N_9) + \sum_{i=1}^{9} C_i^9 N_i + E_9 &= 0,
\end{aligned} \tag{50}
$$

where the summation is made with respect to odd numbers.

Multiply each of the five equations in (50) by l_1, l_3, l_5, l_7, and l_9, respectively, and add together; we have then

$$\begin{aligned} &D(l_1N_1 + l_3N_3 + l_5N_5 + l_7N_7 + l_9N_9) \\ &+ (C_1^1l_1 + C_1^3l_3 + C_1^5l_5 + C_1^7l_7 + C_1^9l_9)\, N_1 \\ &+ (C_3^1l_1 + C_3^3l_3 + C_3^5l_5 + C_3^7l_7 + C_3^9l_9)\, N_3 \\ &+ (C_5^1l_1 + C_5^3l_3 + C_5^5l_5 + C_5^7l_7 + C_5^9l_9)\, N_5 \\ &+ (C_7^1l_1 + C_7^3l_3 + C_7^5l_5 + C_7^7l_7 + C_7^9l_9)\, N_7 \\ &+ (C_9^1l_1 + C_9^3l_3 + C_9^5l_5 + C_9^7l_7 + C_9^9l_9)\, N_9 \\ &+ (l_1E_1 + l_3E_3 + l_5E_5 + l_7E_7 + l_9E_9) = 0. \end{aligned} \tag{51}$$

Now let

$$\begin{aligned} C_1^1l_1 + C_1^3l_3 + C_1^5l_5 + C_1^7l_7 + C_1^9l_9 &= l_1\xi, \\ C_3^1l_1 + C_3^3l_3 + C_3^5l_5 + C_3^7l_7 + C_3^9l_9 &= l_3\xi, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &\ldots, \\ C_9^1l_1 + C_9^3l_3 + C_9^5l_5 + C_9^7l_7 + C_9^9l_9 &= l_9\xi, \end{aligned} \tag{52}$$

and eliminate $l_1, l_3, \ldots, l_9$; we have then

$$\begin{vmatrix} C_1^1-\xi & C_1^3 & C_1^5 & C_1^7 & C_1^9 \\ C_3^1 & C_3^3-\xi & C_3^5 & C_3^7 & C_3^9 \\ C_5^1 & C_5^3 & C_5^5-\xi & C_5^7 & C_5^9 \\ C_7^1 & C_7^3 & C_7^5 & C_7^7-\xi & C_7^9 \\ C_9^1 & C_9^3 & C_9^5 & C_9^7 & C_9^9-\xi \end{vmatrix} = 0. \tag{53}$$

This equation has five real roots. Let them be ξ_1, ξ_3, ξ_5, ξ_7, and ξ_9 and, corresponding to them, equations (52) will give five sets of $l_1, l_3, \ldots, l_9$, or

$$\begin{aligned} &\xi_1:\ l_1^1,\ l_3^1,\ l_5^1,\ l_7^1,\ l_9^1; \\ &\xi_3:\ l_1^3,\ l_3^3,\ l_5^3,\ l_7^3,\ l_9^3; \\ &\xi_5:\ l_1^5,\ l_3^5,\ l_5^5,\ l_7^5,\ l_9^5; \\ &\xi_7:\ l_1^7,\ l_3^7,\ l_5^7,\ l_7^7,\ l_9^7; \\ &\xi_9:\ l_1^9,\ l_3^9,\ l_5^9,\ l_7^9,\ l_9^9. \end{aligned} \tag{54}$$

These five sets of roots and multipliers l's will give Y_m and F_m as

$$\begin{aligned} Y_m &= l_1^mN_1 + l_3^mN_3 + l_5^mN_5 + l_7^mN_7 + l_9^mN_9; \\ F_m &= l_1^mE_1 + l_3^mE_3 + l_5^mE_5 + l_7^mE_7 + l_9^mE_9 \\ &\qquad (m = 1, 3, 5, 7, 9) \end{aligned} \tag{55}$$

and the corresponding five equations

$$D(Y_m) + \xi_mY_m + E_m = 0 \qquad (m = 1, 3, 5, 7, 9) \tag{56}$$

for $Y_1, Y_3, \ldots, Y_9$. They are no longer simultaneous, and are not difficult to solve. The same, of course, applies to even sets, too.

Practically, the equation corresponding to (53) is

$$\begin{bmatrix} 1-\zeta & +8 & -6 & +4 & -2 \\ -3 & 20-\zeta & +18 & -12 & +6 \\ +5 & -15 & 69-\zeta & +20 & -10 \\ -7 & +21 & -35 & 148-\zeta & +14 \\ +9 & -27 & +45 & -63 & 257-\zeta \end{bmatrix} = 0$$

or

$$210830400 - 84156468\zeta + 4625225\zeta^2 - 78771\zeta^3 + 495\zeta^4 - \zeta^5 = 0 \qquad (57)$$

where

$$\zeta = \xi/(4.38649084 \times \tfrac{9}{11}) \qquad (58)$$

and the five roots of (57) are

$$\begin{aligned} \zeta_1 &= 2.9641943, \\ \zeta_3 &= 26.7072817, \\ \zeta_5 &= 74.3718483, \\ \zeta_7 &= 146.456904, \\ \zeta_9 &= 244.499771. \end{aligned} \qquad (59)$$

For even series we have, corresponding to (57), (58), and (59),

$$\begin{bmatrix} 7-\gamma & +16 & -12 & +8 & -4 \\ -8 & 43-\gamma & +24 & -16 & +8 \\ +12 & -24 & 111-\gamma & +24 & -12 \\ -16 & +32 & -48 & 211-\gamma & +16 \\ +20 & -40 & +60 & -80 & 343-\gamma \end{bmatrix} = 0$$

or

$$5217079023 - 593115237\gamma + 16671798\gamma^2 - 172458\gamma^3 + 715\gamma^4 - \gamma^5 = 0 \qquad (60)$$

where

$$\gamma = \xi/(4.38649084 \times 3/4) \qquad (61)$$

and

$$\begin{aligned} \gamma_2 &= 12.8567852, \\ \gamma_4 &= 51.494636, \\ \gamma_6 &= 116.1536052, \\ \gamma_8 &= 207.4231320, \\ \gamma_{10} &= 327.071842. \end{aligned} \qquad (62)$$

Differential Equations for Y's and Their Solutions

From the numerical computations described above, the numerical coefficients of the differential equations for $Y_m(\lambda; \eta)$ were determined as follows:

$$\begin{aligned} D(Y_1) + 11514116\eta^2 Y_1 - 0.4037686 \times 10^{20} &= 0, \\ D(Y_3) + 103741765\eta^2 Y_3 - 6.3254121 \times 10^{20} &= 0, \\ D(Y_5) + 288890008\eta^2 Y_5 - 204.1449265 \times 10^{20} &= 0, \\ D(Y_7) + 568897198\eta^2 Y_7 + 311.4127463 \times 10^{20} &= 0, \\ D(Y_9) + 949734910\eta^2 Y_9 + 257.7413034 \times 10^{20} &= 0 \end{aligned} \qquad (63)$$

and

$$\begin{aligned} &D(Y_2) + 51883042\eta^2 Y_2 + 7.4929828 \times 10^{20} = 0, \\ &D(Y_4) + 207804544\eta^2 Y_4 - 5.3393918 \times 10^{20} = 0, \\ &D(Y_6) + 468733228\eta^2 Y_6 + 258.843806 \times 10^{20} = 0, \\ &D(Y_8) + 837047752\eta^2 Y_8 + 158.8661390 \times 10^{20} = 0, \\ &D(Y_{10}) + 1319885336\eta^2 Y_{10} + 290.043938 \times 10^{20} = 0 \end{aligned} \tag{64}$$

where

$$D = \frac{d^4}{d\lambda^4} - 1082323\eta^2 \frac{d^2}{d\lambda^2} - 14997110 \frac{d}{d\lambda}$$

and

$Y_1, Y_3, \ldots, Y_9$; $Y_2, Y_4, \ldots, Y_{10}$ stand for the following expressions in terms of N_m:

$$\begin{aligned} Y_1 &= N_1 + 0.2167031N_3 - 0.0302211N_5 + 0.0100338N_7 - 0.0045539N_9, \\ Y_3 &= N_1 + 4.6332707N_3 + 1.6778105N_5 - 0.2720794N_7 + 0.1018516N_9, \\ Y_5 &= N_1 - 7.3461297N_3 - 31.5044500N_5 - 13.1812666N_7 + 2.0803739N_9, \\ Y_7 &= N_1 - 4.2883468N_3 + 20.4004807N_5 + 99.0498144N_7 + 47.0163475N_9, \\ Y_9 &= N_1 - 3.6583547N_3 + 9.7849522N_5 - 40.0969663N_7 - 245.9320935N_9 \end{aligned} \tag{65}$$

and

$$\begin{aligned} Y_2 &= N_2 + 0.3172231N_4 - 0.0501698N_6 + 0.0180624N_8 - 0.0086497N_{10}, \\ Y_4 &= N_2 + 4.1262613N_4 + 1.5874177N_6 - 0.2597861N_8 + 0.0995609N_{10}, \\ Y_6 &= N_2 - 5.4990870N_4 - 23.8405548N_6 - 10.3129949N_8 + 1.6109253N_{10}, \\ Y_8 &= N_2 - 3.0874234N_4 + 13.8285447N_6 + 68.6295877N_8 + 33.3180647N_{10}, \\ Y_{10} &= N_2 - 2.5712654N_4 + 6.4271377N_6 - 25.6939046N_8 - 160.9722444N_{10}. \end{aligned} \tag{66}$$

If the 10 functions $Y_1, Y_2, \ldots, Y_{10}$ can be determined by solving the differential equations (63) and (64), it will be possible to compute $N_1(\lambda; \eta)$, $N_2(\lambda; \eta)$, . . . , $N_{10}(\lambda; \eta)$ from the following expressions which are the reversions of the expressions (65) and (66).

$$\begin{aligned} N_1 = {} & +1.0650221\,Y_1 - 0.05878696\,Y_3 - 0.00491292\,Y_5 \\ & - 0.001038174\,Y_7 - 0.000284101\,Y_9, \\ N_3 = {} & -0.2793058\,Y_1 + 0.2586992\,Y_3 + 0.01644431\,Y_5 \\ & + 0.003283144\,Y_7 + 0.000879072\,Y_9, \\ N_5 = {} & +0.1254434\,Y_1 - 0.07711532\,Y_3 - 0.04050495\,Y_5 \\ & - 0.006251173\,Y_7 - 0.001571970\,Y_9, \\ N_7 = {} & -0.05969848\,Y_1 + 0.03367969\,Y_3 + 0.01083299\,Y_5 \\ & + 0.01265900\,Y_7 + 0.002526788\,Y_9, \\ N_9 = {} & +0.02320969\,Y_1 - 0.01264167\,Y_3 - 0.003642393\,Y_5 \\ & - 0.002365708\,Y_7 - 0.004554906\,Y_9 \end{aligned} \tag{67}$$

and

$$\begin{aligned} N_2 = {} & +1.1172140\,Y_2 - 0.1028957\,Y_4 - 0.0109283\,Y_6 \\ & - 0.0026156\,Y_8 - 0.0007745\,Y_{10}, \\ N_4 = {} & -0.3385977\,Y_2 + 0.3066091\,Y_4 + 0.0247985\,Y_6 \\ & + 0.0055792\,Y_8 + 0.0016107\,Y_{10}, \\ N_6 = {} & +0.1613919\,Y_2 - 0.0943220\,Y_4 - 0.0548464\,Y_6 \\ & - 0.0096171\,Y_8 - 0.0026064\,Y_{10}, \\ N_8 = {} & -0.0792995\,Y_2 + 0.0420747\,Y_4 + 0.0147581\,Y_6 \\ & + 0.0184664\,Y_8 + 0.0040001\,Y_{10}, \\ N_{10} = {} & +0.0314504\,Y_2 - 0.0160186\,Y_4 - 0.0050095\,Y_6 \\ & - 0.0034369\,Y_8 - 0.0069853\,Y_{10}. \end{aligned} \tag{68}$$

Substituting the functions $N_1(x;\eta)$, $N_2(x;\eta)$, . . . , $N_{10}(x;\eta)$ thus obtained in (30), we shall obtain the expression for $\psi_1(x, y; \eta)$. Further substitution in (29) will give the solution of the problem as

$$\psi(x, y, z) = \frac{1}{2D_z}\sum_m M_m(y) \cdot \int_0^\infty N_m(x;\eta)\cos\left(\frac{\pi z}{2D_z}\eta\right)d\eta. \tag{69}$$

The method of solving equations (63) and (64) and the evaluation of the integral:

$$\int_0^\infty N_m(x;\eta)\cos\left(\frac{\pi z}{2D_z}\eta\right)d\eta$$

will be discussed in the following section.

Integration of the Differential Equations

The next step will be to solve the 10 differential equations (63) and (64). Let any one of these equations be

$$\frac{d^4Y}{d\lambda^4} - 1082323\eta^2\frac{d^2Y}{d\lambda^2} - 14997110\frac{dY}{d\lambda} + b\eta^2Y + c = 0 \tag{70}$$

where b and c are constants assigned to each of these 10 equations. Since a particular solution of this equation is

$$Y_p = -\frac{c/b}{\eta^2}, \tag{71}$$

the general integral of (70) will be of the form:

$$Y(\lambda) = Ac^{\alpha\lambda} + Bc^{\beta\lambda} + Cc^{\gamma\lambda} + Dc^{\delta\lambda} - \frac{c/b}{\eta^2} \tag{72}$$

where α, β, γ, and δ are the four roots of the algebraic equation:

$$\sigma^4 - 1082323\eta^2\sigma^2 - 14997110\sigma + b\eta^2 = 0 \tag{73}$$

and A, B, C, D are constants to be determined according to the conditions:

$$Y(0) = Y(1) = 0; \qquad Y_1'(0) = Y_1'(1) = 0. \tag{74}$$

The equation (73) has four roots α, β, γ, and δ for any given value of η, and A, B, C, and D all depend upon α, β, γ, and δ.

The parameter η varies from 0 to ∞, and the values of α, β, γ, δ all depend upon η. When η is very large, these four roots are approximately

$$\begin{aligned}\alpha &= +1040.3075\eta,\\ \beta &= -1040.3075\eta,\\ \gamma &= +(b/1082323)^{\frac{1}{2}},\\ \delta &= -(b/1082323)^{\frac{1}{2}}.\end{aligned} \tag{75}$$

As η decreases, α, β, γ, and δ also change gradually. For η less than a certain value between $\eta = 0.4$ and $\eta = 0.3$, β and δ become complex conjugate. As η approaches 0, α, β, and δ approach finite values, while γ decreases indefinitely as $\propto \eta^2$. Thus we have, when $\eta \to 0$,

$$\begin{aligned}\alpha &= 2p,\\ \beta &= -p + iq,\\ \gamma &= b\eta^2/8p^3,\\ \delta &= -p - iq\end{aligned} \tag{76}$$

where

$$p = (14997110)^{\frac{1}{3}},$$

$$q = \frac{\sqrt{3}}{2} p. \tag{77}$$

For the intermediate value of η, these four roots vary continuously except β and δ which change from complex conjugates to real as η increases from $\eta = 0.3$ to $\eta = 0.4$.

Of course, there are 10 series of such four roots of 10 equations (70), each varying with the parameter η.

The constants A, B, C, and D can be, of course, expressed in terms of α, β, γ, and δ. Thus the solution Y becomes, when β and δ are complex conjugates of the form

$$\beta = +p + qi;\ \delta = p - qi, \tag{78}$$

$$\begin{aligned} Y = \Big\{ 1 &+ \frac{\gamma}{\alpha - \gamma} e^{\alpha(\lambda-1)} - \frac{\alpha}{\alpha - \gamma} e^{\gamma(\lambda-1)} + \left(\frac{\alpha}{\alpha - \gamma} e^{-\gamma} - 1 \right) e^{-p\lambda} \cos q\lambda \\ &+ \left(\frac{\alpha}{\alpha - \gamma} e^{-\gamma} \cdot \frac{p + \gamma}{q} - \frac{p}{q} \right) e^{-p\lambda} \sin q\lambda \Big\} \cdot Y_p \end{aligned} \tag{79}$$

and, when α, β, γ, δ are all real,

$$\begin{aligned} Y = \Big[1 &+ \frac{\gamma}{\alpha - \gamma} e^{\alpha(\lambda-1)} + \left\{ \frac{\delta}{\beta - \delta} + \frac{\alpha}{\alpha - \gamma} \cdot \frac{\gamma - \delta}{\beta - \delta} e^{-\gamma} \right\} \cdot e^{\beta\lambda} \\ &- \frac{\alpha}{\alpha - \gamma} e^{\gamma(\lambda-1)} + \left\{ -\frac{\beta}{\beta - \delta} + \frac{\alpha}{\alpha - \gamma} \cdot \frac{\beta - \gamma}{\beta - \delta} e^{-\gamma} \right\} \cdot e^{\rho\lambda} \Big] \cdot Y_p, \end{aligned} \tag{80}$$

where Y_p is the particular solution given by (71), of the equation (70).

When η increases, γ also increases. If we can neglect $e^{-\gamma}$, the expression (80) will be further simplified, and we have

$$Y = \left\{ 1 + \frac{\gamma}{\alpha - \gamma} e^{\alpha(\lambda-1)} + \frac{\delta}{\beta - \delta} e^{\beta\lambda} - \frac{\alpha}{\alpha - \gamma} e^{\gamma(\lambda-1)} - \frac{\beta}{\beta - \delta} e^{\delta\lambda} \right\} \cdot Y_p. \tag{81}$$

Since Y_p is given by (71) as

$$Y_p = -\frac{(c/b)}{\eta^2},$$

the solutions of the equations (63) and (64) will tend to zero as η increases indefinitely.

The values of the roots α, β, γ, and δ of each of the 10 equations given by (63) and (64) were computed numerically and are given in Tables 5–14.

Computation of the Currents

In order to calculate the distribution of the currents at various levels, we had first to compute $Y_1(\lambda;\eta)$, $Y_2(\lambda;\eta)$, . . . , $Y_{10}(\lambda;\eta)$ according to one of the expressions (79), (80), and (81) for

$$\lambda = 0.0000, 0.0025, 0.0050, \ldots, 0.0500, 0.0550, 0.0600, \ldots, 0.1000, 0.1100, 0.1200, 0.1300, \ldots, 0.2000$$

and for

$$\eta = 0.0, 0.1, 0.2, \ldots, 1.0, 2.0, 3.0, \ldots, 10.0.$$

For larger values of η, we have

$$\begin{aligned} \alpha &= +1040.3075\eta, \\ \beta &= -1040.3075\eta, \\ \gamma &= +(b/1082323)^{\frac{1}{2}}, \\ \delta &= -(b/1082323)^{\frac{1}{2}} \end{aligned} \tag{82}$$

very accurately, while $Y_p(\eta)$ are all very small, so that $Y_2(\lambda;\ \eta)$ will be approximately given by

$$Y = \left\{1 - e^{\gamma(\lambda-1)} - e^{-\gamma\lambda}\right\} \cdot \frac{(c/b)}{\eta^2} \tag{83}$$

where γ, and c are independent of η. It will be more convenient to leave (83) as it stands rather than to compute their values against η.

These values of the 10 functions $Y_m(\lambda;\ \eta)$ may be, then, converted into the functions $N_m(\lambda;\ \eta)$ by virtue of formulas (67) and (68).

Substitutions of the functions $N_1(\lambda;\ \eta)$, $N_2(\lambda;\ \eta)$, $N_3(\lambda;\ \eta)$, . . . , $N_{10}(\lambda;\ \eta)$ into equation (69) give the complete solution as

$$\psi(x, y, z) = \frac{1}{2D_z} \sum_{m=1}^{10} M_m(y) \cdot \int_0^\infty N_m(\lambda;\ \eta) \cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta. \tag{84}$$

To evaluate the integral

$$\int_0^\infty N_m(\lambda;\ \eta) \cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta,$$

we have computed the functions $N_m(\lambda;\ \eta)$ for

$$\eta = 0.0,\ 0.1,\ 0.2,\ \ldots,\ 1.0,\ 2.0,\ 3.0,\ \ldots,\ 10.0$$

and the process has to be carried out numerically between $\eta = 0$ and $\eta = 10.0$. For larger values of η, we may use the approximate formula (83).

Let the values of a function $F(\eta)$ for $\eta = 0$, h, and $2h$ be F_0, F_1, and F_2, respectively. Then the interpolation formula in this interval of η will be given by

$$F(\eta) = F_0 + \frac{-3F_0 + 4F_1 - F_2}{2}\left(\frac{\eta}{h}\right) + \frac{F_0 - 2F_1 + F_2}{2}\left(\frac{\eta}{h}\right)^2. \tag{85}$$

Then we have

$$\int_0^{2h} F(\eta) \cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta = F_0 \int_0^{2h} \cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta + \frac{-3F_0 + 4F_1 - F_2}{2h}$$
$$\int_0^{2h} \eta \cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta + \frac{F_0 - 2F_1 + F_2}{2h^2} \int_0^{2h} \eta^2 \cos\left(\frac{\pi z}{2D_z}\right) \eta\, d\eta$$

or

$$\int_0^{2h} F(\eta) \cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta = (a_1 - 3b_1 + c_1) \cdot F_0 + (4b_1 - 2c_1) \cdot F_1 + (b_1 + c_1) \cdot F_2$$

where

$$a_1 = \int_0^{2h} \cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta$$

$$= \frac{1}{\left(\frac{\pi z}{2D_z}\right)} \sin\left(\frac{\pi z}{2D_z} \cdot 2h\right)$$

$$b_1 = \frac{1}{2h}\int_0^{2h} \eta \cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta$$

$$= \frac{1}{\left(\frac{\pi z}{2D_z}\right)} \cdot \frac{\left(\frac{\pi z}{2D_z} \cdot 2h\right)\sin\left(\frac{\pi z}{2D_z} \cdot 2h\right) + \cos\left(\frac{\pi z}{2D_z} \cdot 2h\right) - 1}{\left(\frac{\pi z}{2D_z} \cdot 2h\right)} \tag{86}$$

$$c_1 = \frac{1}{2h^2}\int_0^{2h} \eta^2 \cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta$$

$$= \frac{1}{\left(\frac{\pi z}{2D_z}\right)} \cdot \frac{2\left(\frac{\pi z}{2D_z} \cdot 2h\right)\cos\left(\frac{\pi z}{2D_z} \cdot 2h\right) + \left\{\left(\frac{\pi z}{2D_z} \cdot 2h\right) - 2\right\}\sin\left(\frac{\pi z}{2D_z} \cdot 2h\right)}{\left(\frac{\pi z}{2D_z} \cdot 2h\right)^2}.$$

For larger values of η, we may express $F(\eta)$ in the form

$$F(\eta) = F_0 + \frac{F_{+1} - F_{-1}}{2h}(\eta - \eta_0) + \frac{F_{+1} - 2F_0 + F_{-1}}{2h^2}(\eta - \eta_0)^2$$

where F_{-1}, F_0, and F_{+1} are the values of $F(\eta)$ for $\eta_0 - h$, η_0, and $\eta_0 + h$, respectively.

We have then

$$\int_{\eta_0-h}^{\eta_0+h} F(\eta)\cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta = (-b_2 + c_2)F_{-1} + (a_2 - 2c_2)F_0 + (b_2 + c_2)F_{+1} \tag{87}$$

where

$$a_2 = \left\{1 - \tfrac{1}{6}\left(\frac{\pi z}{2D_z} \cdot h\right)^2\right\}\cos\left(\frac{\pi z}{2D_z}\eta_0\right) \cdot 2h,$$

$$b_2 = \left\{\tfrac{1}{3}\left(\frac{\pi z}{2D_z} \cdot h\right) - \tfrac{1}{30}\left(\frac{\pi z}{2D_z} \cdot h\right)^3\right\}\sin\left(\frac{\pi z}{2D_z}\eta_0\right) h, \tag{88}$$

$$c_2 = \left\{\tfrac{1}{3} - \tfrac{1}{10}\left(\frac{\pi z}{2D_z} \cdot h\right)^2\right\}\cos\left(\frac{\pi z}{2D_z}\eta_0\right) \cdot h.$$

The integration was carried out taking $h = 0.1$ in the interval $0 \leqq \eta \leqq 1.0$ and $h = 1$ for the interval $1.0 \leqq \eta \leqq 10.0$. For larger values of η, integration was carried out by using formula (82) which is only inversely proportional to η^2. In this case, we have only to evaluate the integral:

$$\int_{10.0}^{\infty} \frac{1}{\eta^2}\cos\left(\frac{\pi z}{2D_z}\eta\right) d\eta. \tag{89}$$

TABLE 2

COMPUTED VELOCITY* OF THE KUROSHIO AT DIFFERENT DEPTHS $z = 0$, $\frac{1}{2} D_z$, D_z, $\frac{3}{2} D_z$, $2D_z$, AND $3D_z$ ALONG THE 33° N PARALLEL, ASSUMING $D_z = 75$ METERS

Distance from west boundary (km.)	0	25	50	75	100	125	150	175	200	225	250	275
Surface velocity (cm/sec)	0	163	217	196	146	94	52	25	12	7	8	11
Velocity at $z = \frac{1}{2} D_z$ (cm/sec)	0	150	206	186	135	84	43	17	4	0	1	4
Velocity at $z = D_z$ (cm/sec)	0	138	188	167	120	72	32	7	−5	−8	−6	−3
Velocity at $z = \frac{3}{2} D_z$ (cm/sec)		123	168	149	104	47	21	9	−12	−14	−12	−8
Velocity at $z = 2 D_z$ (cm/sec)	0	107	147	129	87	44	11	−9	−22	−19	−15	−11
Velocity at $z = 3 D_z$ (cm/sec)	0	79	110	94	59	24	−3	−25	−25	−17	−18	−13

* To get the velocity for values of D_z other than 75 m. multiply these velocities by $75/D_z$, where D_z is expressed in meters.

The computations were rather tedious and took three computers more than 6 months to complete for the surface, $\frac{1}{2}D_z$, D_z, $\frac{3}{2}D_z$, $2D_z$, and $3D_z$. The values of α, β, γ, and δ, which are the roots of the quadratic equations (73) with η as a parameter and computed for the necessary values of η, are compiled in Tables 5–14.

The values of the stream-function were computed for the westernmost one-fifth part of the entire expanse of the ocean, and the streamlines were drawn for the layers $z = 0$ (surface), $\frac{1}{2}D_z$, D_z, $\frac{3}{2}D_z$, $2D_z$, and $3D_z$. The computations were not carried out for the deeper levels and for the part to the east of this area, partly because we did not have enough time to compute, and partly because the central and eastern part are not as interesting. We have only a very slow zonal flow in the central part and very diffuse meridional flow close to the eastern coast of the ocean. Indeed, the California and Peru Currents are considered to be produced by local winds as proved by Munk (1950).

The circulation patterns in the area close to the western coast were obtained from these computations and illustrated in Figures 3A–F. The discussions for them will be given in the following paragraphs.

Table 2 gives the velocities of the western current in the subtropic gyre corresponding to the Kuroshio, or the Japan Current, at the depths $z = 0$ (surface), $\frac{1}{2}D_z$, D_z, $\frac{3}{2}D_z$, $2D_z$, and $3D_z$ along the 33° N parallel which is the swiftest part of this mighty current. These velocities were computed by the formula:

$$v = \left(\frac{\partial\psi}{\partial x}\right)_{\lambda_0} = \frac{\psi(\lambda_0 + \Delta\lambda) - \psi(\lambda_0 - \Delta\lambda)}{\text{linear distance of } 2\Delta\lambda},$$

and taking $D_z = 75$ meters. Because the velocity is inversely proportional to the quantity D_z, we can compute it for any other value of D_z. For this we have only to multiply these figures by $75/D_z$ where D_z is expressed in meters. The maximum surface velocity of 217 cm/sec agrees with actual observations very closely.

Table 3 gives the distribution of E-W components along a meridian 24° of longitude east of the western boundary, or one fifth of the entire east-west expanse of the Pacific Ocean off the western coast. At this distance from the western coast, the coastal effect nearly vanishes and the pattern of the circulation consists of approximately zonal flows. In this table the value of D_z was again assumed to be 75 meters. Discussions concerning these results will be made in the following paragraphs.

Surface Circulation

The numerical result for the horizontal circulation has been worked out for several levels specified by the ratio z/D_z. We show here those of the surface ($z = 0$), $z/D_z = 0.5$,

$z/D_z = 1$, $z/D_z = 1.5$, $z/D_z = 2$, and $z/D_z = 3$. The most important of them is, of course, the surface circulation, and Figure 3A shows its pattern. The gross features of the current distribution on the surface thus do not seem to differ much from those given by Munk (1950) and by the author (1951) for the distribution of mass transport streamlines.

TABLE 3

ZONAL DISTRIBUTION OF E-W COMPONENTS OF OCEAN CURRENTS EXPRESSED IN CM/SEC ASSUMING D_z = 75 METERS (+ EASTWARD, − WESTWARD)

LATITUDE	$z = 0$ (SURFACE)	$z = \frac{1}{2} D_z$	$z = D_z$	$z = \frac{3}{2} D_z$	$z = 2 D_z$	$z = 3 D_z$	REMARKS
60° N	0	0	0	0	0	0	Arctic
57° N	0	− 1.2	− 2.1	− 2.7	− 2.9	− 2.8	Current
54° N	+ 2.5	+ 0.2	− 1.7	− 2.8	− 3.5	− 3.6	
51° N	+ 8.2	+ 5.0	+ 2.3	+ 0.5	− 0.7	− 1.7	
48° N	+15.3	+11.6	+ 3.6	+ 6.1	+ 4.2	+ 2.3	West
45° N	+20.4	+16.7	+13.3	+10.7	+ 8.7	+ 5.9	Wind
42° N	+20.6	+17.4	+14.5	+12.2	+10.3	+ 7.3	Drift
39° N	+14.9	+12.8	+10.8	+ 9.2	+ 7.8	+ 5.7	
36° N	+ 5.0	+ 4.3	+ 3.6	+ 3.0	+ 2.6	+ 2.0	
33° N	− 5.8	− 5.0	− 4.4	− 3.8	− 3.3	− 2.3	
30° N	−14.1	−12.1	−10.3	− 8.8	− 7.4	− 5.3	North
27° N	−18.3	−15.4	−12.8	−10.7	− 8.8	− 6.0	Equatorial
24° N	−18.4	−15.1	−12.2	− 9.8	− 7.9	− 5.1	Current
21° N	−15.8	−12.5	− 9.8	− 7.5	− 5.8	− 3.5	
18° N	−11.8	− 8.9	− 6.5	− 4.6	− 3.3	− 1.7	
15° N	− 7.3	− 5.0	− 3.2	− 1.9	− 1.1	− 0.4	
12° N	− 2.7	− 1.2	0	+ 0.7	+ 1.0	+ 0.8	
9° N	+ 2.4	+ 3.3	+ 3.9	+ 4.0	+ 3.8	+ 2.8	Equatorial
6° N	+ 7.5	+ 7.0	+ 9.3	+ 7.1	+ 6.5	+ 4.8	Counter
3° N	+ 7.6	+ 8.2	+ 8.4	+ 8.2	+ 7.5	+ 5.7	Current
0°	+ 5.3	+ 6.3	+ 7.0	+ 7.0	+ 6.7	+ 5.2	
3° S	− 0.7	+ 1.6	+ 2.5	+ 3.2	+ 3.5	+ 3.1	
6° S	− 6.9	− 6.4	− 4.1	− 2.5	− 1.5	− 0.5	
9° S	−16.4	−12.9	− 9.8	− 7.5	− 5.8	− 3.5	South
12° S	−22.9	−16.8	−13.3	− 4.9	− 8.3	− 5.2	Equatorial
15° S	−21.5	−17.6	−14.1	−11.4	− 9.1	− 5.9	Current
18° S	−18.3	−14.9	−11.9	− 9.5	− 7.6	− 4.8	
21° S	−12.6	−10.2	− 8.0	− 6.3	− 5.0	− 3.0	
24° S	− 5.9	− 4.8	− 3.7	+ 3.0	− 2.3	− 1.4	
27° S	+ 1.2	+ 0.8	+ 0.6	+ 0.4	+ 0.3	0	
30° S	+ 8.2	+ 6.5	+ 5.0	+ 3.9	+ 3.0	+ 1.7	
33° S	+14.9	+12.0	+ 9.5	+ 7.5	+ 5.8	+ 3.5	West
36° S	+20.0	+16.5	+13.2	+10.6	+ 8.4	+ 5.3	Wind
39° S	+22.4	+18.4	+14.8	+12.0	+ 9.7	+ 6.3	Drift
42° S	+20.0	+16.4	+13.1	+12.5	+ 8.4	+ 5.6	
45° S	+13.7	+10.6	+ 7.9	+ 6.0	+ 4.7	+ 3.0	
48° S	+ 5.0	+ 2.8	+ 1.0	0	− 0.5	− 0.6	
51° S	− 2.6	− 3.4	− 4.8	− 5.1	− 4.9	− 3.8	Antarctic
54° S	− 6.2	− 6.8	− 7.2	− 8.0	− 6.4	− 4.8	Current
57° S	− 4.8	− 5.0	− 5.1	− 4.9	− 4.4	− 3.3	
60° S	0	0	0	0	0	0	

Because of the considerable labor contained in the calculation, the computation is confined only to the western part of the ocean bounded by two meridians, $\lambda = 0$ and $\lambda = 0.2$, that is, 24° of longitude apart. Choice of the western part of the ocean for the computation was made because the circulation patterns in that section are more complicated and hence more interesting. In the central part of the ocean we will have indeed a very slow motion approximately in east-west direction, while very diffuse meridional motion will exist close to the eastern coast.

We have a number of gyres in the surface circulation corresponding to those obtained in Munk's (1950) and the author's (1951) results with respect to mass transport. We have a broad gyre with strong western current flowing north in the latitudes between 20° and 40° N and corresponding to the Kuroshio, or Japan Current. We also notice one boundary vortex, but the secondary boundary vortex is not distinct. We have a subtropic gyre with the western current flowing south. This corresponds to the Mindanao Current. Of course, we have a faint subarctic gyre corresponding to the Oyashio, or the Kurile Current.

On the surface of the Southern Pacific Ocean, we have western currents flowing north a little south of the equator and in the subantarctic latitudes. Between these two we have a strong current corresponding to the East Australian Current, though actually this current never develops so strongly because of many passages connecting the Southern Pacific to the Indian Ocean through the numerous islands and archipelagoes in the Australian-Asiatic Mediterranean. Had we not these passages together with the Southern Antarctic Circumpolar Ocean, we could have a much stronger western current in the South Pacific Ocean than actually observed.

It looks also rather strange that we do not have any strong westward flow in the latitudes between 5° N and 2° S although actually the northern margin of the South Equatorial Current is in this zone. This is because the Equatorial Counter Current appears in our theoretical result much broader and much more diffuse than actually observed. This is also the same in Munk's and Hidaka's results. The theory of the Equatorial Counter Current has been attacked and explained by several authors (Montgomery and Palmén, 1940; Neumann, 1947) in some other ways than ours.

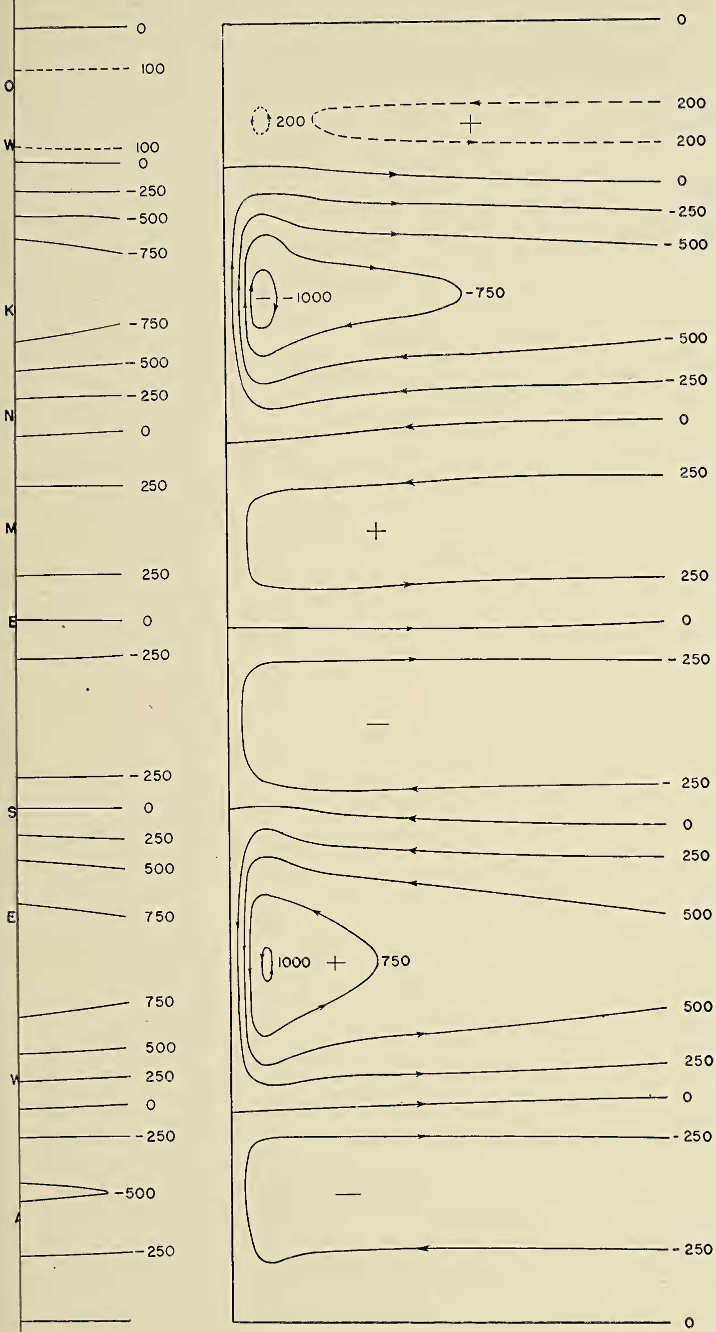
Evaluation of the Coefficient of Vertical Mixing

The streamlines in Figure 3A are drawn for an interval $\Delta\psi = 250 \times 10^{10}/D_z$ cm²/sec of the stream function. The velocity can be determined as the ratio $\Delta\psi/\Delta x$, where Δx is the actual distance between two consecutive streamlines. Because these diagrams are not drawn in a common scale for the north-south and east-west directions, it would be rather laborious to compute the magnitude of current velocity for all parts of the Pacific. Still it will be easy to determine when the streamlines run in exactly north-south or east-west directions.

The values of the stream function at several points along the 33° N parallel are computed as compiled in Table 4. Assuming the Pacific Ocean is 10,000 kilometers across in its east-west direction, we obtained the velocity of the Kuroshio at its swiftest zone, which is located approximately 55 km. off the coast, to be 329 cm/sec, 219 cm/sec, 165 cm/sec, and 110 cm/sec according as we assume D_z = 50 m., 75 m., 100 m., and 150 m., respectively.

Actual velocity of the Kuroshio has been estimated at approximately 3 to 5 knots, or about 150 to 250 cm/sec in its swiftest zone. From Table 4 we recognize that the computed velocity of the Kuroshio, assuming for D_z a value between 50 m. and 150 m., agrees with

$\frac{Z}{D_z} = 2.0$



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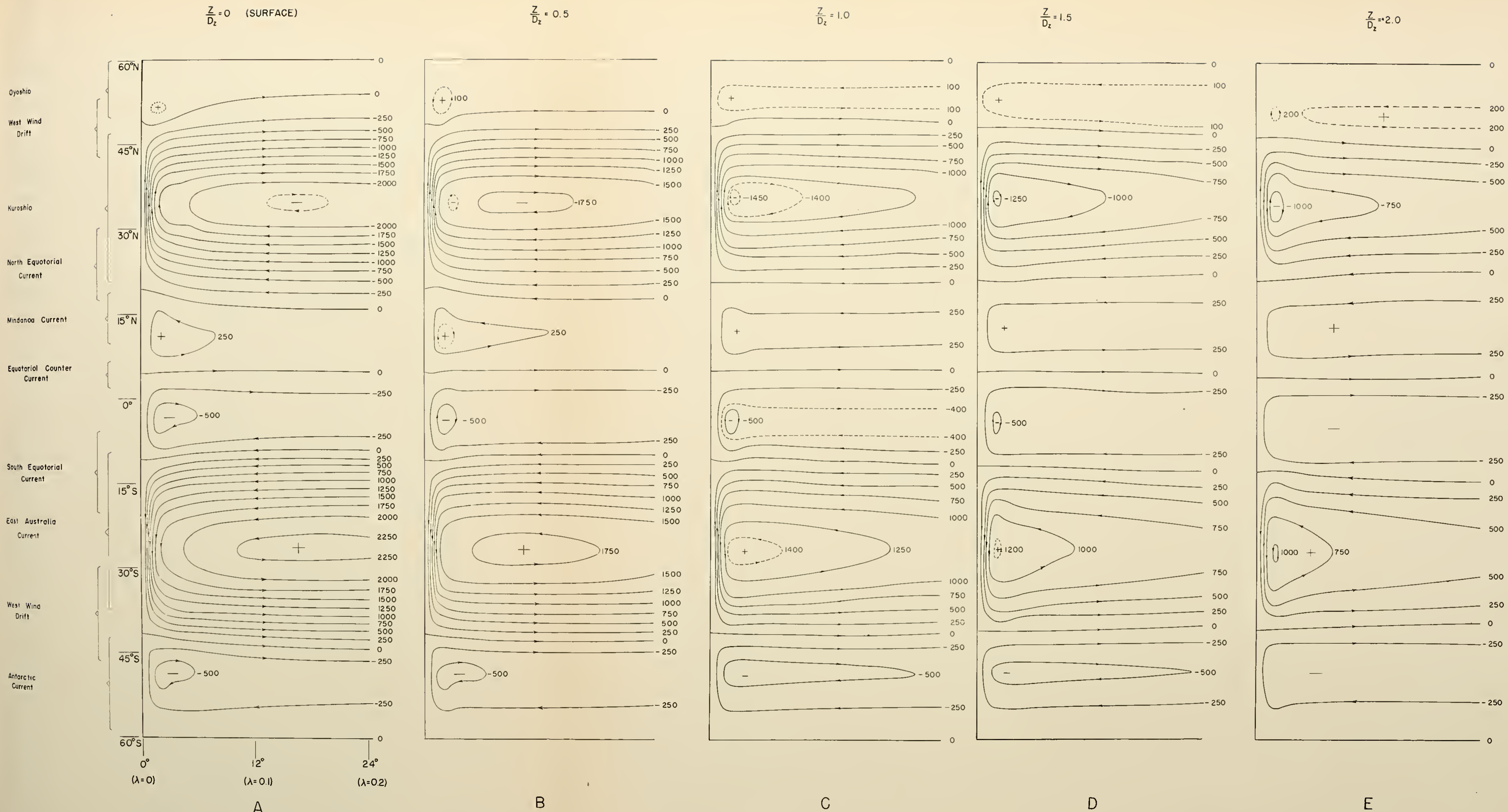

FIG. 3. Computed circulation in the western part of the Pacific Ocean at six levels. *a*, Surface; *b*, $z = \frac{1}{2} D_z$; *c*, $z = 1.0\, D_z$; *d*, $z = 1.5\, D_z$; *e*, $z = 2\, D_z$; *f*, $z = 3.0\, D_z$.

TABLE 4
COMPUTATION OF THEORETICAL SURFACE VELOCITY OF THE KUROSHIO ACROSS THE 33° N PARALLEL

λ =	0	.0025	.0050	.0075	.0100	.0125	.0150	.0175	.0200	.0225	.0250	.0275	
Distance from west boundary (km.) =	0	25	50	75	100	125	150	175	200	225	250	275	
ψ =	0	612	815	735	546	352	195	95	44	28	31	41	$\times 10^{10}$ cm²/sec / D_z
$\Delta\psi/\Delta x$ =	0	122.4	163.0	157.0	109.2	70.4	39.0	19.0	8.8	5.6	6.2	8.2	$\times 10^{4}$ cm/sec / D_z
Computed Velocity (cm/sec) For D_z = 50 m.	0	245	326	314	218	141	78	38	10	11	12	16	
For D_z = 75 m.	0	163	217	207	146	94	52	25	12	7	8	11	
For D_z = 100 m.	0	122	163	157	109	70	39	19	9	6	6	8	
For D_z = 150 m.	0	82	109	105	73	37	26	13	6	4	4	5	

the observed values fairly well. The previously determined values for D_z fall mostly in this range also. This enables us to compute the values of vertical coefficient of mixing from formula (18). The above values of D_z correspond to the values 188, 422, 750, and 1688 g/cm/sec of A_z, respectively. These are, of course, values consistent with the results derived from many other different sources. (Sverdrup, *et al.*, 1942.)

Subsurface Circulation

Figure 3B, C, D, E, and F show the horizontal distribution of streamlines in the level $\frac{1}{2}D_z$, D_z, $1\frac{1}{2}D_z$, $2D_z$, and $3D_z$ below the sea surface, respectively. All give patterns similar to the sea surface circulation shown in Figure 3A. We have western currents and a boundary vortex attached to each gyre. The only difference noticed is a general subsidence of the motion as we go down into deeper layers. Still, we see that the intensity of motion is only reduced to as low as half that of the sea surface even in the layer $3D_z$. Figure 4 shows the comparison of the current velocity profiles along the 33° N parallel at several levels to that on the surface of the sea assuming D_z = 75 m. The maximum speeds are seen at about 55 kilometers off the western boundary. Although the Japanese Islands are not disposed parallel to a meridian, the above result agrees with the observed profiles of this mighty current quite satisfactorily. Another result of particular interest is that, at a distance greater than about 150 km., there is a flow to the south with much greater velocity than in upper layers. This counter current reaches a maximum speed of 20 to 30 cm/sec at about 200 km. off the western coast, despite the practically motionless upper layers. Figure 5 gives the comparison of the zonal distribution of E-W components along a meridian 24 degrees of longitude to the east off the western boundary. In this longitude it is expected that the influence of the western boundary nearly vanishes and the actual flow pattern of the Pacific circulation is disposed mostly as a zonal current system. The velocities of the current in these diagrams were computed assuming D_z = 75 meters. For computing the velocities when the value of D_z is different, we have only to multiply these figures by $75/D_z$, where D_z is expressed in meters. If we assume, however, that the value D_z = 75 m. is consistent, we have for the maximum surface velocities of the North Pacific Current, North Equatorial Current, Equatorial Counter Current, South Equatorial Current, and Antarctic Circumpolar Current 22, 19, 8, 23, and 23 cm/sec, respectively. They are reduced to 18, 16, 8, 18, and 18 cm/sec, respectively, at a level $\frac{1}{2}D_z$ and to 11, 9, 8,

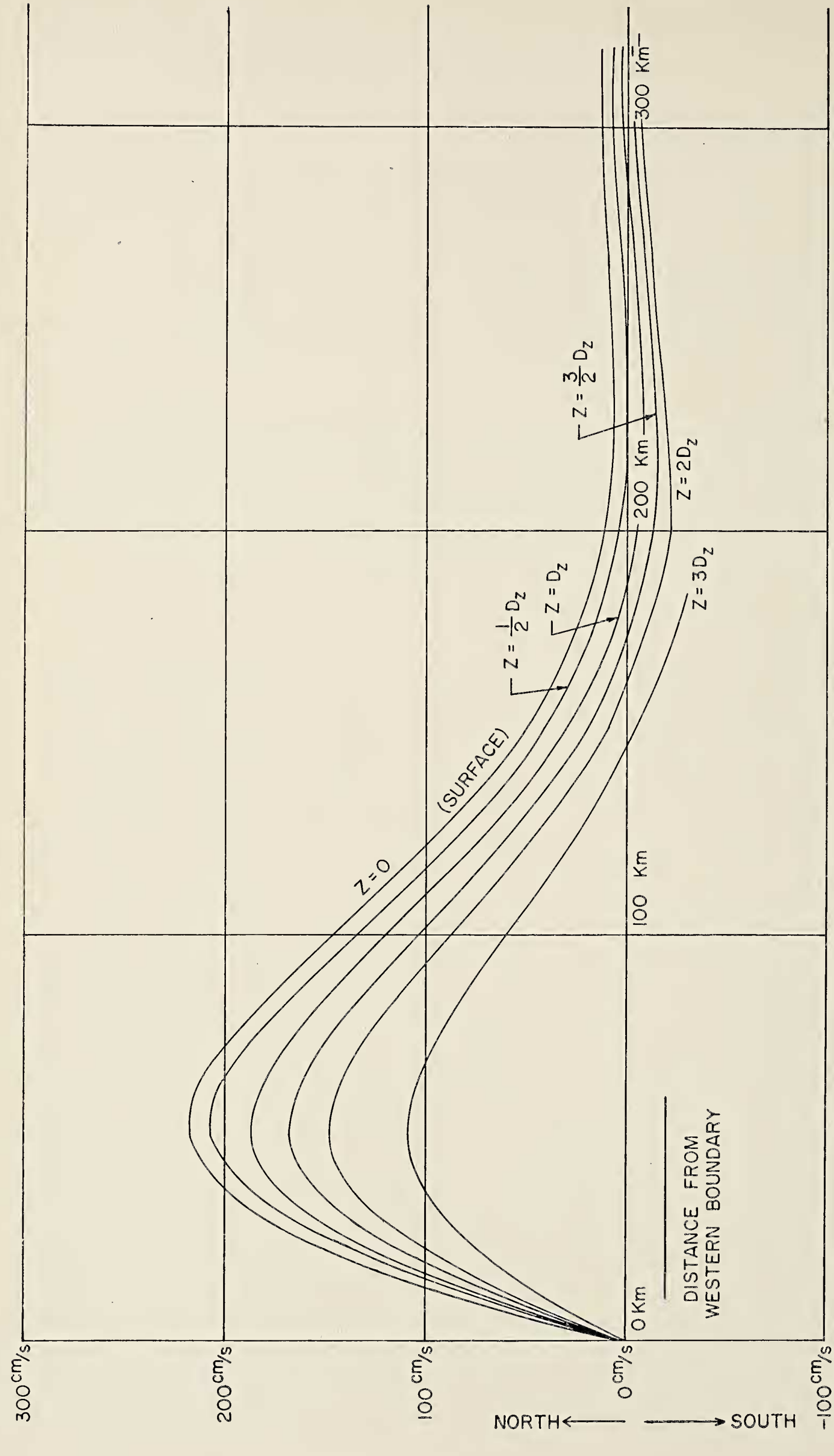

FIG. 4. Theoretical velocity profiles of the Kuroshio at several different depths, along the 33° N parallel, assuming $D_z = 75$ meters. (To obtain profiles for D_z other than 75 meters, multiply these velocities by $75/D_z$, where D_z is expressed in meters.)

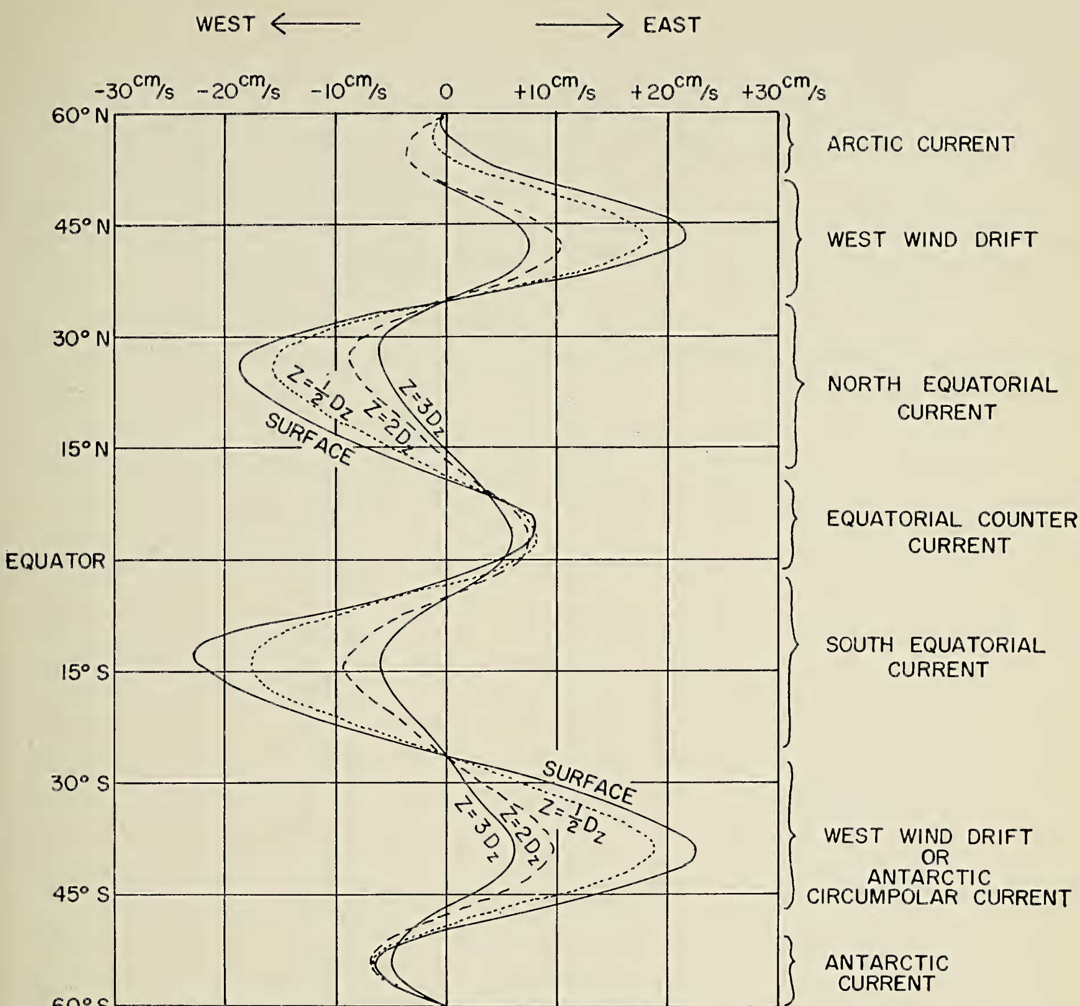

FIG. 5. Zonal distribution of E-W components of ocean currents at several depths expressed in cm/sec, when $D_z = 75$ meters. (To obtain the velocity for other values of D_z, multiply these velocities by $75/D_z$, where D_z is expressed in meters.)

9, and 10, respectively, at $2D_z$. The Equatorial Counter Current remains nearly unaltered in its speed in all depths compared above.

Vertical Variation of the Currents

The most important objective of the present research is to get a certain idea about the vertical structure of the wind-driven circulation in the Pacific Ocean. This will be, of course, impossible to obtain from former theories which have been mostly propounded with respect to the mass transport.

As the problem is three-dimensional, the numerical computation is rather tedious. For this reason the author has not yet been able to finish the computation below the level

$z = 3D_z$. Still, we should be able to expect some important conclusions from what has been completed thus far.

First of all it is very interesting that the wind-driven currents exist in a layer much deeper than that expected from the classical theory of Ekman (1905). According to Ekman's theory, a wind-driven current is confined to the surface layer about D_z thick, and we can expect practically no drift current at a deeper level than 150 m. except very close to the equator. From our computation, it can be shown that the current velocity does not drop as low as half the surface value even at a level $3D_z$. If we take $D_z = 75$ meters, this depth is 225 meters.

This conclusion will help us understand the fact that wind-driven currents can penetrate into a layer several hundred meters deep, several times as deep as that expected

TABLE 5

FOUR ROOTS OF THE EQUATION: $\sigma^4 - 1082323\eta^2\sigma^2 - 14997110\sigma + 11514116\eta^2 = 0$ ($m = 1$) FOR DIFFERENT VALUES OF η. REFER TO EQUATION (73)

η	α	β	γ	δ
0.1	261.21657	− 130.612124 − 200.881495i	0.0076775	−130.612124 +200.881495i
0.2	304.26551	− 152.148109 − 161.695419i	0.0307072	−152.148109 +161.695419i
0.3	371.21622	− 185.642643 − 77.168515i	0.0690668	−185.642643 + 77.168515i
0.4	454.08159	− 363.12143	0.122667	− 91.082828
0.5	545.93222	− 489.85023	0.191279	− 56.273270
0.6	642.62244	− 603.98223	0.274436	− 38.914645
0.7	741.98396	− 713.66160	0.371324	− 28.693687
0.8	842.89309	− 821.22710	0.480691	− 22.146678
0.9	944.74621	− 927.63340	0.600783	− 17.713594
1.0	1047.20274	− 1033.34356	0.72937	− 14.58855
1.1	1150.06112	− 1138.60821	0.86382	− 12.31673
1.2	1253.19657	− 1243.57336	1.00136	− 10.62457
1.3	1356.52895	− 1348.32950	1.13922	− 9.33867
1.4	1460.00495	− 1452.93512	1.27490	− 8.34473
1.5	1563.5883	− 1557.4297	1.40631	− 7.56490
1.6	1667.2526	− 1661.8399	1.53190	− 6.94465
1.7	1770.9803	− 1766.1856	1.65059	− 6.44526
1.8	1874.7576	− 1870.4809	1.76177	− 6.03849
1.9	1978.5740	− 1974.7356	1.86521	− 5.70361
2.0	2082.4224	− 2078.9583	1.96096	− 5.42509
2.5	2601.9746	− 2599.7576	2.33637	− 4.55339
3.0	3121.8104	− 3120.2708	2.58146	− 4.12107
3.5	3641.7804	− 3640.6492	2.74475	− 3.87598
4.0	4161.8219	− 4160.9559	2.85725	− 3.72328
4.5	4681.9050	− 4681.2207	2.93742	− 3.62168
5.0	5202.0138	− 5201.4595	2.99627	− 3.55053
5.5	5722.1396	− 5721.6815	3.04065	− 3.49871
6.0	6242.2768	− 6241.8919	3.07487	− 3.45977
6.5	6762.4223	− 6762.0943	3.10178	− 3.42975
7.0	7282.5734	− 7282.2906	3.12332	− 3.40610
7.5	7802.7290	− 7802.4827	3.14080	− 3.38714
8.0	8322.8879	− 8322.6714	3.15518	− 3.37170
8.5	8843.0494	− 8842.8577	3.16716	− 3.35890
9.0	9363.2128	− 9363.0417	3.17724	− 3.34830
9.5	9883.3778	− 9883.2242	3.18579	− 3.33937
10.0	10403.544	−10403.405	3.19310	− 3.33166

from Ekman's theory of the limit of the wind-driven currents. This implies that the motion of water in most parts of the *oceanic troposphere* could be produced by the stresses of the permanent wind system prevailing over the oceans. In other words, the winds are responsible not only for the currents in the skin layer of the ocean, but also for most of the circulation in the oceanic troposphere.

We have long considered that the winds are responsible only for the current motion in the surface layer of about 200 meters. This depth is nothing but the "depth of the frictional influence" defined by Ekman. To explain the circulation in deeper parts of the troposphere, we had to assume a very strong convection current and slope current. Still we had a distinct difference in the circulation patterns between the troposphere and stratosphere. These circumstances have made several problems very complicated. Defant (1928) defined the troposphere as the part of the ocean in which we can expect strong currents due to violent turbulence and convection. Still we can have violent convection in the seas of higher latitudes beyond the polar fronts which are no longer defined as troposphere. The conclusion that the drift currents penetrate into much deeper layers than D_z is much in favor of the definition of the troposphere as the upper layer of the ocean in which strong currents are present.

The explanation of the result that we can have a strong motion even in a layer several hundred meters deep might seem to be possible by assuming slope currents which would be produced as the effect of purely wind-driven water masses piled up against the land barriers. As a matter of fact, Ekman's theory assumes no boundaries and a constant latitude. We can prove the existence of slope current in an ocean having boundaries partly or completely enclosing it. The slope current is uniform from the surface down to the bottom. This fact seems in favor of the theoretical result we have obtained. Still, we must give attention to the fact that the velocity of slope current is always inversely proportional to the depth of the sea. When the depth is large, as we see in the actual oceans, the slope current will not be strong enough to account for those large velocities we have obtained at the depth twice or three times as large as D_z.

We do not know an appropriate explanation of the theoretical result that the effect of winds can be felt at a depth several times as large as Ekman's depth of frictional influence. It is to be hoped that someone may be able to solve this question satisfactorily in the near future.

SUMMARY

(1) A theory of the general circulation of water in the Pacific Ocean produced by the semipermanent wind system prevailing over this ocean is propounded.

(2) The velocity is used to express the water motion which has formerly been explained by several authors in terms of mass transport.

(3) The Pacific Ocean is considered to be a rectangular ocean extending from 60° S to 60° N latitudes and from 0° to 120° longitude, and a zonal distribution of the wind system determined from actual observations has been assumed.

(4) The effects of horizontal turbulence and the meridional variation of the Coriolis force have been taken into account.

(5) The patterns of horizontal circulation are obtained in terms of streamlines for the sea surface and several deeper layers specified by the ratio z/D_z where z is the geometrical depth below the surface and D_z the depth of the frictional influence, a measure of vertical turbulence.

(6) Surface circulation has a pattern similar to that actually observed and does not differ much from Munk's result obtained in terms of mass transport. We have very strong western currents and boundary vortices.

(7) The magnitude of the Kuroshio and other western currents was computed from the distribution of the streamlines in each level. The velocity is inversely proportional to D_z so that we can determine it by assuming an appropriate value for D_z. A value of D_z between 50 m. and 150 m. gives values most reasonable and consistent with the actual observations.

(8) Subsurface circulations also show similar patterns except for a general decrease in motion as we go down into deeper layers. Still, it is remarkable that we have far stronger currents than expected from Ekman's classical theory even at a depth much larger than Ekman's depth of frictional influence at which we can scarcely expect any motion. This seems to show us that the winds are responsible for most of the tropospheric motion of water.

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TABLE 6

FOUR ROOTS OF THE EQUATION: $\sigma^4 - 1082323\eta^2\sigma^2 - 14997110\sigma + 51883042\eta^2 = 0$ ($m = 2$) FOR DIFFERENT VALUES OF η. REFER TO EQUATION (73)

η	α	β	γ	δ
0.1	261.2089	$-$ 130.6217 $-$ 200.8820i	0.034595	$-$130.6217 $+$200.8820i
0.2	304.2429	$-$ 152.1906 $-$ 161.7127i	0.138326	$-$152.1906 $+$161.7127i
0.3	371.1853	$-$ 185.7480 $-$ 77.3462i	0.310731	$-$185.7480 $+$ 77.3462i
0.4	454.0497	$-$ 363.0414	0.55003	$-$ 91.55829
0.5	545.9024	$-$ 489.8041	0.85179	$-$ 56.95000
0.6	642.5958	$-$ 603.9481	1.20755	$-$ 39.85528
0.7	741.9602	$-$ 713.6338	1.60417	$-$ 29.93054
0.8	842.8718	$-$ 821.2038	2.02475	$-$ 23.69276
0.9	944.7270	$-$ 927.6127	2.45104	$-$ 19.56531
1.0	1047.1851	$-$ 1033.3250	2.86653	$-$ 16.72663
1.1	1150.0451	$-$ 1138.5916	3.25872	$-$ 14.71229
1.2	1253.1819	$-$ 1243.5582	3.61994	$-$ 13.24361
1.3	1356.5153	$-$ 1348.3155	3.94678	$-$ 12.14657
1.4	1459.9923	$-$ 1452.9222	4.23898	$-$ 11.30907
1.5	1563.5762	$-$ 1557.4175	4.49829	$-$ 10.65703
1.6	1667.2415	$-$ 1661.8286	4.72746	$-$ 10.14036
1.7	1770.9698	$-$ 1766.1750	4.92960	$-$ 9.72442
1.8	1874.7475	$-$ 1870.4707	5.10800	$-$ 9.38487
1.9	1978.5647	$-$ 1974.7263	5.26555	$-$ 9.10440
2.0	2082.4135	$-$ 2078.9493	5.40496	$-$ 8.86915
2.5	2601.9675	$-$ 2599.7504	5.90334	$-$ 8.12046
3.0	3121.8036	$-$ 3120.2640	6.19651	$-$ 7.73611
3.5	3641.7746	$-$ 3640.6434	6.38114	$-$ 7.51234
4.0	4161.8168	$-$ 4160.9508	6.50416	$-$ 7.37016
4.5	4681.9004	$-$ 4681.2161	6.58996	$-$ 7.27426
5.0	5202.0097	$-$ 5201.4554	6.65206	$-$ 7.20632
5.5	5722.1359	$-$ 5721.6778	6.69840	$-$ 7.15650
6.0	6242.2734	$-$ 6241.8885	6.73386	$-$ 7.11876
6.5	6762.4187	$-$ 6762.0907	6.76160	$-$ 7.08960
7.0	7282.5705	$-$ 7282.2877	6.78369	$-$ 7.06649
7.5	7802.7263	$-$ 7802.4800	6.80157	$-$ 7.04787
8.0	8322.8854	$-$ 8322.6689	6.81623	$-$ 7.03274
8.5	8843.0470	$-$ 8842.8553	6.82841	$-$ 7.02011
9.0	9363.2108	$-$ 9363.0397	6.83861	$-$ 7.00973
9.5	9883.376	$-$ 9883.222	6.84727	$-$ 7.00089
10.0	10403.542	$-$10403.404	6.85474	$-$ 6.99330

TABLE 7

FOUR ROOTS OF THE EQUATION: $\sigma^4 - 1082323\eta^2\sigma^2 - 14997110\sigma + 103741765\eta^2 = 0$ ($m = 3$) FOR DIFFERENT VALUES OF η. REFER TO EQUATION (73)

η	α	β	γ	δ
0.1	261.19836	-130.63377 $-200.88184i$	0.06917	-130.63377 $+200.88184i$
0.2	304.21377	-152.24513 $-161.73511i$	0.27648	-152.24513 $+161.73511i$
0.3	371.14543	-185.88275 $-77.57694i$	0.62007	-185.88275 $+77.57694i$
0.4	454.0086	− 362.9384	1.09300	− 92.16317
0.5	545.8643	− 489.7452	1.67853	− 57.79759
0.6	642.5616	− 603.9042	2.34715	− 41.00454
0.7	741.9296	− 713.5981	3.05872	− 31.39024
0.8	842.8444	− 821.1731	3.77053	− 25.44186
0.9	944.7023	− 927.5862	4.44709	− 21.56323
1.0	1047.1631	− 1033.3018	5.06562	− 18.92693
1.1	1150.0246	− 1138.5702	5.61601	− 17.07041
1.2	1253.1630	− 1243.5387	6.09746	− 15.72172
1.3	1356.4978	− 1348.2976	6.51453	− 14.71474
1.4	1459.9760	− 1452.9056	6.87418	− 13.94458
1.5	1563.5610	− 1557.4020	7.18401	− 13.34300
1.6	1667.2272	− 1661.8141	7.45125	− 12.86434
1.7	1770.9563	− 1766.1614	7.68234	− 12.47727
1.8	1874.7348	− 1870.4579	7.88287	− 12.15980
1.9	1978.5526	− 1974.7141	8.05757	− 11.89611
2.0	2082.4020	− 2078.9377	8.21039	− 11.67465
2.5	2601.9583	− 2599.7412	8.74444	− 10.96153
3.0	3121.7968	− 3120.2572	9.05081	− 10.59044
3.5	3641.7687	− 3640.6376	9.24114	− 10.37229
4.0	4161.8116	− 4160.9456	9.36693	− 10.23297
4.5	4681.8958	− 4681.2115	9.45421	− 10.13849
5.0	5202.0056	− 5201.4513	9.51716	− 10.07142
5.5	5722.1322	− 5721.6741	9.56401	− 10.02208
6.0	6242.2700	− 6241.8851	9.59981	− 9.98471
6.5	6762.4151	− 6762.6871	9.62776	− 9.95572
7.0	7282.5676	− 7282.2848	9.64999	− 9.93278
7.5	7802.7236	− 7802.4773	9.66797	− 9.91430
8.0	8322.8828	− 8322.6663	9.68271	− 9.89921
8.5	8843.0446	− 8842.8528	9.69494	− 9.88672
9.0	9363.2083	− 9363.0372	9.70520	− 9.87627
9.5	9883.3736	− 9883.2201	9.71389	− 9.86743
10.0	10403.5401	−10403.4015	9.72132	− 9.85989

TABLE 8

FOUR ROOTS OF THE EQUATION: $\sigma^4 - 1082323\eta^2\sigma^2 - 14997110\sigma + 207804544\eta^2 = 0$ ($m = 4$) FOR DIFFERENT VALUES OF η. REFER TO EQUATION (73)

η	α	β	γ	δ
0.1	261.17780	− 130.658175 − 200.886400i	0.138550	−130.658175 +200.886400i
0.2	304.15535	− 152.354359 − 161.784857i	0.553368	−152.354359 +161.784857i
0.3	371.06547	− 186.151299 − 78.030786i	1.237127	−186.151299 + 78.030786i
0.4	453.9263	− 362.7315	2.162989	− 93.35781
0.5	545.7878	− 489.6268	3.271038	− 59.43202
0.6	642.4929	− 603.8161	4.469333	− 43.14513
0.7	741.8683	− 713.5264	5.657710	− 33.99958
0.8	842.7895	− 821.1121	6.758457	− 28.43583
0.9	944.6527	− 927.5328	7.73047	− 24.85037
1.0	1047.1177	− 1033.2539	8.56385	− 22.42764
1.1	1149.9834	− 1138.5273	9.26719	− 20.72327
1.2	1253.1250	− 1243.4993	9.85663	− 19.48229
1.3	1356.4627	− 1348.2616	10.35076	− 18.55185
1.4	1459.9433	− 1452.8723	10.76555	− 17.83659
1.5	1563.5304	− 1557.3709	11.11539	− 17.27486
1.6	1667.1985	− 1661.7851	11.41206	− 16.82546
1.7	1770.9293	− 1766.1340	11.66511	− 16.46041
1.8	1874.7092	− 1870.4321	11.88225	− 16.15935
1.9	1978.5283	− 1974.6896	12.06966	− 15.90836
2.0	2082.3790	− 2078.9146	12.23233	− 15.69674
2.5	2601.9398	− 2599.7227	12.79226	− 15.00941
3.0	3121.7831	− 3120.2435	13.10803	− 14.64763
3.5	3641.7570	− 3640.6258	13.30241	− 14.43361
4.0	4161.8014	− 4160.9354	13.43018	− 14.29618
4.5	4681.8867	− 4681.2024	13.51850	− 14.20280
5.0	5201.9974	− 5201.4431	13.58205	− 14.13631
5.5	5722.1247	− 5721.6666	13.62926	− 14.08736
6.0	6242.2632	− 6241.8783	13.66528	− 14.05018
6.5	6762.4092	− 6762.0812	13.69337	− 14.02137
7.0	7282.5617	− 7282.2789	13.71571	− 13.99851
7.5	7802.7181	− 7802.4718	13.73376	− 13.98006
8.0	8322.8777	− 8322.6612	13.74855	− 13.96505
8.5	8843.0398	− 8842.8480	13.76081	− 13.95261
9.0	9363.2031	− 9363.0320	13.77110	− 13.94218
9.5	9883.369	− 9883.215	13.77982	− 13.93334
10.0	10403.535	−10403.397	13.78725	− 13.92583

TABLE 9

Four Roots of the Equation: $\sigma^4 - 1082323\eta^2\sigma^2 - 14997110\sigma + 288890008\eta^2 = 0$ ($m = 5$) For Different Values of η. Refer to Equation (73)

η	α	β	γ	δ
0.1	261.16177	$-130.677185 - 200.89077i$	0.19260	$-130.677185 + 200.89077i$
0.2	304.10979	$-152.43931 - 161.82152i$	0.76882	$-152.43931 + 161.82152i$
0.3	371.00312	$-186.35885 - 78.38164i$	1.71458	$-186.35885 + 78.38164i$
0.4	453.8619	− 362.5695	2.97958	− 94.27196
0.5	545.7281	− 489.5344	4.45733	− 60.65104
0.6	642.4394	− 603.7475	5.99960	− 44.69154
0.7	741.8205	− 713.4706	7.46727	− 35.81721
0.8	842.7467	− 821.0646	8.77347	− 30.45556
0.9	944.6141	− 927.4913	9.88811	− 27.01095
1.0	1047.0828	− 1033.2171	10.81805	− 24.68375
1.1	1149.9513	− 1138.4939	11.58649	− 23.04389
1.2	1253.0955	− 1243.4691	12.22046	− 21.84683
1.3	1356.4353	− 1348.2335	12.74497	− 20.94673
1.4	1459.9178	− 1452.8463	13.18123	− 20.25277
1.5	1563.5066	− 1557.3468	13.54645	− 19.70629
1.6	1667.1763	− 1661.7625	13.85429	− 19.26805
1.7	1770.9083	− 1766.1128	14.11561	− 18.91107
1.8	1874.6893	− 1870.4120	14.33896	− 18.61630
1.9	1978.5095	− 1974.6706	14.53110	− 18.36997
2.0	2082.3610	− 2078.8964	14.69740	− 18.16196
2.5	2601.9254	− 2599.7082	15.26688	− 17.48409
3.0	3121.7694	− 3120.2297	15.58610	− 17.12578
3.5	3641.7452	− 3640.6140	15.78195	− 16.91313
4.0	4161.7911	− 4160.9251	15.91042	− 16.77647
4.5	4681.8775	− 4681.1932	15.99912	− 16.68342
5.0	5201.9892	− 5201.4349	16.06288	− 16.61715
5.5	5722.1172	− 5721.6591	16.11022	− 16.56832
6.0	6242.2563	− 6241.8714	16.14639	− 16.53126
6.5	6762.4032	− 6762.0752	16.17447	− 16.50244
7.0	7282.5558	− 7282.2730	16.19684	− 16.47963
7.5	7802.7126	− 7802.4663	16.21491	− 16.46125
8.0	8322.8725	− 8322.6560	16.22972	− 16.44623
8.5	8843.0349	− 8842.8431	16.24200	− 16.43379
9.0	9363.1991	− 9363.0280	16.25230	− 16.42337
9.5	9883.3649	− 9883.2114	16.26102	− 16.41455
10.0	10403.5319	−10403.3933	16.26847	− 16.40704

TABLE 10

FOUR ROOTS OF THE EQUATION: $\sigma^4 - 1082323\eta^2\sigma^2 - 14997110\sigma + 468733228\eta^2 = 0$ ($m = 6$) FOR DIFFERENT VALUES OF η. REFER TO EQUATION (73)

η	α	β	γ	δ
0.1	261.1262	− 130.71934 − 200.89311i	0.312479	−130.71934 +200.89311i
0.2	304.0087	− 152.62721 − 161.90576i	1.245716	−152.62721 +161.90576i
0.3	370.8647	− 186.81402 − 79.14854i	2.763347	−186.81402 + 79.14854i
0.4	453.7192	− 362.2092	4.74125	− 96.25127
0.5	545.5957	− 489.3292	6.94392	− 63.21045
0.6	642.3205	− 603.5949	9.10051	− 47.82611
0.7	741.7145	− 713.3466	11.02080	− 39.38867
0.8	842.6516	− 820.9590	12.63321	− 34.32580
0.9	944.5284	− 927.3991	13.94741	− 31.07676
1.0	1047.0047	− 1033.1347	15.00642	− 28.87640
1.1	1149.8801	− 1138.4198	15.85925	− 27.31956
1.2	1253.0299	− 1243.4015	16.54939	− 26.17782
1.3	1356.3746	− 1348.1714	17.11211	− 25.31536
1.4	1459.8613	− 1452.7887	17.57493	− 24.64758
1.5	1563.4537	− 1557.2930	17.95898	− 24.11967
1.6	1667.1265	− 1661.7121	18.28046	− 23.69487
1.7	1770.8615	− 1766.0655	18.55182	− 23.34779
1.8	1874.6451	− 1870.3674	18.78268	− 23.06042
1.9	1978.4675	− 1974.6283	18.98052	− 22.81973
2.0	2082.3211	− 2078.8563	19.15124	− 22.61604
2.5				
3.0	3121.7487	− 3120.2090	20.05542	− 21.59512
3.5				
4.0	4161.7756	− 4160.9061	20.38232	− 21.25182
4.5				
5.0	5201.9767	− 5201.4224	20.53547	− 21.08974
5.5				
6.0	6242.2460	− 6241.8611	20.61914	− 21.00404
6.5				
7.0	7282.5470	− 7282.2642	20.66976	− 20.95256
7.5				
8.0	8322.8648	− 8322.6483	20.70268	− 20.91919
8.5				
9.0	9363.1903	− 9363.0192	20.72530	− 20.89636
9.5				
10.0	10403.524	−10403.385	20.74146	− 20.88003